

VARIATIONAL FORMULATION OF RELATIVISTIC FLUID THERMODYNAMICS

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ABSTRACT

As a first step toward generalizing reversible thermodynamics from the case of a homogeneous system to that of a system whose local velocity may be a function of its position in space-time, a variational principle is derived for relativistic reversible adiabatic flow of a compressible fluid. This is done by identifying the thermodynamic internal energy function for a given sample of the fluid with its Hamiltonian function, and then invoking the canonical equations of motion. Both in order to bring the rest-mass energy into the formalism, as well as to provide a means of labelling and identifying different samples of fluid, it is necessary to introduce a new thermodynamic variable, which is just the molar initial momentum vector of the fluid sample in question. It turns out that this vector is intimately related to the vorticity of the flow, and if it had been omitted, the formalism would have been implicitly limited to a description of vorticity-free flow.

The Lagrangian density, as seen in the fixed laboratory frame, that results from identifying the Hamiltonian with the thermodynamic internal energy is just the thermodynamic pressure. This must be regarded as a function of the generalized coordinates that are canonical to the particle density, the entropy density, and the initial momentum vector (all regarded as generalized momenta). More precisely, the pressure is a function of the proper-time derivatives of these coordinates. These time derivatives are equal to the molar free enthalpy, the rest-temperature, and the initial velocity respectively. Because, in the laboratory frame, the proper-time derivative of a variable is defined as the contraction of the velocity four-vector with the four-gradient of the variable, the pressure is also a function of the fluid velocity.

This variational principle yields the correct form of the stress-energy tensor for reversible adiabatic flow of a compressible fluid (together with the necessary statements of particle and entropy conservation), and automatically gives the expression for the solution of Euler's equation of motion for the fluid in terms of the four-gradients of the generalized coordinates.

INTRODUCTION

The discussion of this article is in the spirit of well-known attempts¹ to bring continuum mechanics within the framework of thermodynamics by treating local velocity as just one more thermodynamic variable to be taken into account with all the others. The basic approach consists of identifying the appropriate thermal energy function with the Hamiltonian of the system,

and the corresponding canonical equations with the mechanical and thermodynamical equations of motion of the system.

This general approach was first applied to the case of a homogeneous system by Helmholtz² in 1886, and adapted to relativity theory in 1907 by Planck³. Planck's theory was developed before four-dimensional tensor analysis and the modern covariance concept had fully evolved. Consequently, although it was form-invariant under Lorentz transformations, it fell completely outside the framework of tensor analysis, which meant that, for all but the simplest applications, it was completely unworkable. (Reviews of both the early⁴ and recent⁵ history of relativistic thermodynamics are available elsewhere.)

In 1939 Van Dantzig⁶ constructed a manifestly covariant thermodynamics, and applied it to fluids⁷, but his work failed to lift the obscurity surrounding the intimate three-way relation that binds together thermodynamics, fluid dynamics, and the canonical formalism. This relation stems from the fact that, if the right choice of variables is made, the thermodynamic energy density function plays the role of Hamiltonian density, and the thermodynamic pressure plays the role of Lagrangian density.

The identification of pressure with Lagrangian density had already been made in 1908 by Hargreaves⁸ for the case of non-relativistic potential flow. Van Dantzig⁷ generalized this identification to the relativistic case, but, although the point was not explicitly made, his proof was likewise limited to the case of potential flow, because he did not include the variables that are necessary for a completely general description of vorticity. (Others have since given relativistic variational principles that are free of this limitation, but these principles all involve the imposition of constraints, and do not make the identification of the Lagrangian density with the thermodynamic pressure.)

Notation

The analysis will be carried out entirely within the framework of special relativity. Boldface Latin or Greek letters will designate four-vectors, and light-face characters will designate scalars. A superior dot will designate differentiation with respect to proper-time τ , i.e. the time derivative as seen by an observer moving with the fluid. Contraction of two four-vectors will be indicated as the dot product of the corresponding boldface characters. Indices will be explicitly indicated only in the case of two-index tensors, and when indices are indicated, the summation convention will be used.

Intensive thermodynamic quantities, and extensive quantities that are referred to one mole of the fluid, will be designated by capital letters. Thus T and P are temperature and pressure respectively, and V , S , U , H and G are the molar volume, entropy, energy, enthalpy and Gibbs function (free enthalpy) respectively. The number of moles per unit volume is $n \equiv 1/V$. Extensive quantities referred to unit *volume* (not unit *mass*!) of the fixed laboratory frame will be designated by the appropriate lower-case Roman character. For example, $u \equiv nU$ is the internal energy per unit volume in the

laboratory frame. Densities referred to the convected fluid frame that is based on coordinate planes embedded in the fluid and moving with it will be designated by the corresponding primed letter. Thus n' and u' are respectively the molar density and molar energy density referred to the convected frame.

ONE-DIMENSIONAL CANONICAL FORMALISM

From the point of view of an observer who remains stationary with respect to a given sample of fluid and refers all measurements to the convected frame, everything can be described as a function of a single variable—the proper-time τ of the sample of fluid under study. Because the fluid appears to remain at rest, the fluid velocity \mathbf{v} does not enter into such a description. When the canonical formalism derived from such an approach is referred to the fixed laboratory frame, however, proper-time differentiation must be defined as $d/d\tau \equiv \mathbf{v} \cdot \partial$ where ∂ is the four-gradient operator, and this brings \mathbf{v} into the formalism. Thus the development of the one-dimensional canonical formalism referred to the convected fluid frame is the first step in arriving at the desired variational principle referred to the laboratory frame.

In reversible adiabatic flow the molar entropy and the total number of particles in the fluid are conserved quantities. Our approach will consist of expressing these two conservation laws in terms of two scalar constants of motion of the fluid. The internal energy will then be written as a function of these two constants of motion and of the proper time. Identifying the internal energy with the Hamiltonian of the system and the constants of motion with generalized momentum coordinates, we are led to the canonical equations of motion.

In order to arrive at the desired statement of conservation of particles, we first note that the molar rest-volume V (not to be confused with the Lorentz-contracted molar volume $V^* = V/\Gamma$ where $\Gamma \equiv [1 - (v/c)^2]^{-\frac{1}{2}}$) may be written $V = JV'$ where $V' = (V)_{\tau=0}$ is the molar volume referred to the convected frame, which is a constant of motion, and is equal to the initial value of V at $\tau = 0$, and J is the function of τ that describes the time-dependence of V that results from compression or expansion of the fluid. The Lorentz-contracted molar volume is thus $V^* = V/\Gamma = (J/\Gamma)V'$. Because intervals of laboratory-time dt and proper-time $d\tau$ are related by $dt = \Gamma d\tau$ we have:

$$d\mathcal{V} \equiv c dt dV^* = c\Gamma d\tau(J/\Gamma) dV' = J(c d\tau dV') \tag{1}$$

where $d\mathcal{V}$ is the element of four-volume in the laboratory frame and $c d\tau dV'$ is the corresponding four-volume element in the convected frame. Thus J is just the Jacobian of the transformation between laboratory coordinates and convected coordinates.

Using $V = JV'$, the thermodynamic equation $dU = T dS - P dV$ would become

$$dU = T dS - (PJ)dV' - (PV'\dot{J})d\tau \tag{2}$$

where $\dot{S} = \dot{V}' = 0$ and $U = U(S, V', \tau)$ would be the thermodynamic potential that we could identify with the Hamiltonian. However, because

we are dealing with a continuum, it is more appropriate to work with densities rather than with molar quantities. For this reason, we eliminate V' , U and S in favour of n' , u' and s' where

$$n' \equiv 1/V' = J/V = Jn; \quad u' \equiv n'U; \quad s' \equiv n'S \quad (3)$$

Making these substitutions in 2, we find

$$du' = T ds' + G dn' - (PJ) d\tau \quad (4)$$

where $G = U + PV - TS$ is the molar Gibbs function. Thus $u' = u'(s', n', \tau)$ is a function of two constants of motion and of the proper-time.

Before identifying u' with the Hamiltonian of the system, we note that equation 4 has two deficiencies which luckily can both be removed by the addition of a single term. First, from the relativistic point of view, the rest-mass energy density $m'c^2 \equiv n'Mc^2$ (where M is the molar rest-mass) should not be isolated from all other contributions to the energy density. Hence u' should be replaced by the total energy density $\tilde{u}' \equiv n'\tilde{U}$ that includes the rest-mass energy density.

The second deficiency of equation 4 arises from the fact that, if we are to describe a fluid rather than just isolated moles of gas that in no way interact with one another, then we must in some way introduce into the formalism parameters that label and identify each mole of gas and distinguish it from all others. Because these parameters will enter into the formalism, they must have a physical significance that is essential to the description of the fluid. Both of these requirements, labelling and physical significance, are satisfied by the *initial momentum vector* $\mathbf{K} = (M\mathbf{v})_{\tau=0}$ which is the momentum possessed by the mole of gas at $\tau = 0$. In doing this we are effectively postulating that the inability to distinguish between two or more moles of gas that would result if their \mathbf{K} -vectors were all equal, represents a physical degeneracy with observable consequences. (We shall, in fact, see that such a degeneracy corresponds to vorticity-free flow.) Thus \mathbf{K} , like the molar entropy S , is a preserved fossil of the initial conditions of the fluid. The vector \mathbf{K} is normalized to the molar mass M , i.e. $K \equiv (\mathbf{K} \cdot \mathbf{K})^{\frac{1}{2}} = Mc$, and so $M \equiv K/c$ can be used as the definition of molar mass, and \tilde{U} becomes $\tilde{U} \equiv U + c(\mathbf{K} \cdot \mathbf{K})^{\frac{1}{2}}$. There exists an alternative procedure for relating \tilde{U} and U that is not only more general, but also closer to the spirit of thermodynamical formalism. We may regard $\tilde{U} = U(S, n, \mathbf{K})$ as the basic thermodynamic potential and S , n (or V), and \mathbf{K} as the basic variables. We then *define* M as $Mc^2 \equiv (\partial\tilde{U}/\partial\mathbf{K}) \cdot \mathbf{K}$. This definition is consistent with $\tilde{U} = U(S, n) + c(\mathbf{K} \cdot \mathbf{K})^{\frac{1}{2}}$ where $c(\mathbf{K} \cdot \mathbf{K})^{\frac{1}{2}} = Mc^2$, but it is more general, and is applicable regardless of the \mathbf{K} -dependence of \tilde{U} . Using this more general definition of M , we *define* the purely thermal energy function U as:

$$U \equiv \tilde{U} - (\partial\tilde{U}/\partial\mathbf{K}) \cdot \mathbf{K} \equiv \tilde{U} - Mc^2 \quad (5)$$

Although relation 5 represents the most general way of defining U and M , in this paper we shall assume that the \mathbf{K} -dependence of \tilde{U} is given by $c(\mathbf{K} \cdot \mathbf{K})^{\frac{1}{2}} = Mc^2$ where M is a constant parameter. In such a case $\partial\tilde{U}/\partial\mathbf{K} = c\mathbf{K}/K = \mathbf{K}/M \equiv \mathbf{v}$ where $\mathbf{v} \equiv (\mathbf{v})_{\tau=0}$ is the initial velocity of the mole of gas in question at $\tau = 0$.

If $\boldsymbol{\kappa}' \equiv n'\mathbf{K}$ is the initial momentum density referred to the convected frame, the density relation that corresponds to relation 5 is

$$u' \equiv \tilde{u}' - \mathbf{v} \cdot \boldsymbol{\kappa}' \equiv \tilde{u}' - m'c^2 \text{ where } \mathbf{v} \equiv \partial\tilde{u}'/\partial\boldsymbol{\kappa}' = \partial\tilde{U}/\partial\mathbf{K} \quad (6)$$

Thus we see that the definition of u' in terms of \tilde{u}' and $\boldsymbol{\kappa}'$ (or of U in terms of \tilde{U} and \mathbf{K}) amounts to a Legendre transformation that replaces the variable $\boldsymbol{\kappa}'$ (or \mathbf{K}) with $\mathbf{v} \equiv \partial\tilde{u}'/\partial\boldsymbol{\kappa}' = \partial\tilde{U}/\partial\mathbf{K}$.

From expressions 4 and 6 we obtain the basic thermodynamic equation of the fluid

$$d\tilde{u}' = T ds' + G dn' + \mathbf{v} \cdot d\boldsymbol{\kappa}' - (P\dot{J}) d\tau \quad (7)$$

This is to be compared with the well-known expression for the differential of the Hamiltonian $E = E(p, q, \tau)$:

$$\begin{aligned} dE &= \sum_p (\partial E/\partial p)_{q, \tau} dp + \sum_q (\partial E/\partial q)_{p, \tau} dq + (\partial E/\partial \tau)_{p, q} d\tau \\ &= \sum_p \dot{q} dp - \sum_q \dot{p} dq - (\partial L/\partial \tau)_{q, \dot{q}} d\tau \end{aligned} \quad (8)$$

where the Lagrangian L is defined as follows:

$$L = L(\dot{q}, q, \tau) \equiv \sum_p p(\partial E/\partial p)_{q, \tau} - E \quad (9)$$

We now identify \tilde{u}' with E . Note that although $\tilde{u}' = n'\tilde{U}$ is a density, it does in fact represent the energy of a fixed number of particles, namely the number contained in unit volume of the convected frame, and so there is no inconsistency in regarding it as the Hamiltonian of a definite dynamical system.

It turns out that, for consistency, it is necessary to identify the thermodynamic variables s' , n' and $\boldsymbol{\kappa}'$ with generalized momenta, rather than with generalized coordinates. Doing this, and designating the coordinates q that are conjugate to the momenta $p = (s', n', \boldsymbol{\kappa}')$ by $q = (\mathcal{T}, \mathcal{G}, \boldsymbol{\xi})$ respectively, comparison of 7 and 8 yields the following equations:

$$T = \dot{\mathcal{T}}; G = \dot{\mathcal{G}}; \mathbf{v} = \dot{\boldsymbol{\xi}}; P\dot{J} = (\partial L/\partial \tau)_{q, \dot{q}} \quad (10)$$

The fact that \tilde{u}' is independent of the coordinates \mathcal{T} , \mathcal{G} and $\boldsymbol{\xi}$ yields the desired equations of motion:

$$\dot{s}' = \dot{n}' = \dot{\boldsymbol{\kappa}}' = 0, \text{ which implies } \dot{\mathbf{K}} = 0 \quad (11)$$

the last equation resulting from $\boldsymbol{\kappa}' = n'\mathbf{K}$ and $\dot{n}' = \dot{\boldsymbol{\kappa}}' = 0$.

From definition 9 we find:

$$\begin{aligned} L &\equiv \sum_p p(\partial\tilde{u}'/\partial p) - \tilde{u}' = n'(G + ST + \mathbf{K} \cdot \mathbf{v} - \tilde{U}) \\ &= n'(H - U) = n'PV = (n'/n)P = JP \end{aligned} \quad (12)$$

where use has been made of 3 and 5. Since $\dot{J} \equiv dJ/d\tau = \partial J/\partial \tau$, equations 12, together with the last equation of 10, yield $(\partial P/\partial \tau)_{q, \dot{q}} = 0$, which means that $P = P(\dot{\mathcal{T}}, \dot{\mathcal{G}}, \dot{\boldsymbol{\xi}})$ is a function of the generalized velocities $\dot{\mathcal{T}} = T$, $\dot{\mathcal{G}} = G$, $\dot{\boldsymbol{\xi}} = \mathbf{v}$ alone, and not an explicit function of τ . For example, in the case of a perfect gas, for which $P = nRT$, where R is the gas constant and

$\gamma \equiv c_p/c_v = \text{constant}$ is the ratio of specific heats, the functional form of P is

$$P = P_0(T/T_0)^{1/(\gamma-1)} \exp \{ [G + M\mathbf{c}(\mathbf{v} \cdot \mathbf{v})^{1/2}] / RT \} \quad (13)$$

where P_0 and T_0 are constants.

Because $L = JP$, the Lagrangian equations of motion are:

$$d[\partial(JP)/\partial\dot{q}]/d\tau = \partial(JP)/\partial q \quad \text{or} \quad d[J(\partial P/\partial\dot{q})]/d\tau = 0 \quad (14)$$

where use has been made of the fact that P is independent of the qs , and J is an explicit function of τ , being independent of the qs and $\dot{q}s$. To evaluate $\partial P/\partial\dot{q}$ we first note that:

$$P = n(H - U) = n(G + ST + \mathbf{K} \cdot \mathbf{v} - \tilde{U}) = nG + sT + \boldsymbol{\kappa} \cdot \mathbf{v} - \tilde{u} \quad (15)$$

where now the densities are all referred to the laboratory frame. Next we note that, from 5 and the relation $du = T ds + G dn$, we have

$$d\tilde{u} = T ds + G dn + \mathbf{v} \cdot d\boldsymbol{\kappa} \quad (16)$$

Taking the differential form of 15 and using 16, we find:

$$dP = s dT + n dG + \boldsymbol{\kappa} \cdot d\mathbf{v} = s d\mathcal{T} + n d\mathcal{G} + \boldsymbol{\kappa} \cdot d\boldsymbol{\xi} \quad (17)$$

Using 17 to evaluate $\partial P/\partial\dot{q}$, we arrive at the following Lagrangian equations of motion:

$$0 = d(Js)/d\tau = \dot{s}'; \quad 0 = d(Jn)/d\tau = \dot{n}'; \quad 0 = d(J\boldsymbol{\kappa})/d\tau = \dot{\boldsymbol{\kappa}}' \quad (18)$$

These, of course, agree with the canonical equations 11. (If we had identified some or all of the thermodynamic variables with generalized coordinates q , rather than with generalized momenta p , this agreement would not have occurred.)

In the same way that, in arriving at 5 and 6, we noted that the mass density $m' \equiv n'M$ could be defined in terms of the $\boldsymbol{\kappa}'$ -dependence of u' , we now note that the mass density $m \equiv nM$ (referred now to the laboratory frame) can be defined in terms of the $\boldsymbol{\xi}$ -dependence of P :

$$mc^2 \equiv (\partial\tilde{u}/\partial\boldsymbol{\kappa}) \cdot \boldsymbol{\kappa} \equiv \mathbf{v} \cdot \boldsymbol{\kappa} = (\partial P/\partial\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \quad (19)$$

Thus, the definition of the molar mass M may be taken to be:

$$M \equiv n^{-1}(\partial P/\partial\boldsymbol{\xi}) \cdot \boldsymbol{\xi}/c^2 = (\partial P/\partial\mathcal{G})^{-1} (\partial P/\partial\boldsymbol{\xi}) \cdot \boldsymbol{\xi}/c^2 \quad (20)$$

VARIATIONAL PRINCIPLE FOR FLUID

The Lagrangian equations can be obtained from the following variational principle: $0 = \delta \int L d\tau = \delta \int JP d\tau$. This refers to the fluid contained in unit volume of the convected frame. If the integrand were $JP d\tau dV'$, the principle would refer to the sample of fluid contained in the volume dV' . Since the fluid contained in each volume element dV' must individually and independently satisfy the requirement $0 = \delta \int_0^{\tau} JP d\tau dV'$, then it must follow that $0 = \delta \int_{\tau, v'} JP d\tau dV'$ where now the integration extends over V' as well as over τ . Thus, referring to relations 1, we arrive at the following variational principle for the fluid:

$$0 = \delta \int_{\tau, v'} P J c d\tau dv' = \delta \int_{v'} P d\mathcal{V} \quad (21)$$

where $d\mathcal{V} \equiv c dt dx dy dz$ is the four-volume element in the laboratory frame, and \mathcal{V} is the four-volume occupied by the fluid between the specified initial and final times.

Because the proper-time τ is no longer the independent variable, the operation $d/d\tau$ must be defined in terms of the fluid velocity \mathbf{v} as $d/d\tau \equiv \mathbf{v} \cdot \partial$ where ∂ is the four-gradient operator. Because \mathbf{v} must remain normalized ($\mathbf{v} \cdot \mathbf{v} = c^2$) during the variation process, it must be parametrized in some way so that the normalization will be guaranteed. This can be done most conveniently by introducing a vector ρ whose direction is \mathbf{v}

$$\mathbf{v} \equiv c\rho/\rho \quad \text{where} \quad \rho \equiv (\rho \cdot \rho)^{\frac{1}{2}} \quad (22)$$

It will turn out that the norm ρ does not appear in any of the Euler–Lagrange equations resulting from 21. The components of ρ are to be regarded as generalized coordinates, rather than as velocities. Using the definition 22 for \mathbf{v} we have:

$$G \equiv \mathcal{G} \equiv \mathbf{v} \cdot \partial \mathcal{G} = \rho^{-1} \rho \cdot \partial \mathcal{G} \quad (23)$$

Similar expressions define \mathcal{T} and ξ .

The more detailed statement of the variational principle given in 21 is

$$0 = \sum_q \int_{\mathcal{V}} (\delta q) [(\partial P / \partial q) - \partial \cdot \mathbf{p}] d\mathcal{V} + \sum_q \int_{\mathcal{S}} (\delta q) \mathbf{p} \cdot \mathbf{d}\mathcal{S} \quad (24)$$

where $\mathbf{p} \equiv \partial P / \partial (\partial q)$ and $\mathbf{d}\mathcal{S}$ is an element of the hypersurface \mathcal{S} that bounds the four-volume \mathcal{V} over which the integration is carried out. The variational principle is thus equivalent to the requirement that the Euler–Lagrange equations $\partial \cdot \mathbf{p} = \partial P / \partial q$ be satisfied, and that the variables have definitely assigned values on the boundary so that $\delta q = 0$ on \mathcal{S} .

Referring to 17 and 23, the calculation of the generalized momenta can be illustrated by the case for \mathcal{G} :

$$\mathbf{p}_g \equiv \partial P / \partial (\partial \mathcal{G}) = (\partial P / \partial G) [\partial G / \partial (\partial \mathcal{G})] = n\mathbf{v} \quad (25)$$

Similarly we find $\mathbf{p}_k = S n v$, $p_{\xi k}^i = n K_k v^i$, and $p_{\rho k}^j = 0$. Thus the Euler–Lagrange equations corresponding to variation of \mathcal{G} , \mathcal{T} and ξ respectively are:

$$0 = \partial \cdot (\mathbf{v}n) = \partial \cdot (\mathbf{v}nS) = \partial \cdot (\mathbf{v}n\mathbf{K}) \quad (26)$$

Variation of ρ yields the following equation:

$$\begin{aligned} 0 &= \partial P / \partial \rho = \sum_q (\partial P / \partial \dot{q}) [\partial (\mathbf{v} \cdot \partial q) / \partial \rho] \\ &= (c/\rho) \sum_q (\partial P / \partial \dot{q}) [\partial q - \mathbf{v}(\dot{q}/c^2)] \\ &= (cn/\rho) [\partial \mathcal{G} + S \partial \mathcal{T} + (\partial \xi) \cdot \mathbf{K} - (M + H/c^2)\mathbf{v}] \end{aligned} \quad (27)$$

or

$$(M + H/c^2)\mathbf{v} = \partial \mathcal{G} + S \partial \mathcal{T} + (\partial \xi) \cdot \mathbf{K} \quad (28)$$

Equations 26 are just the required conservation laws. Using the first, the second and third could also be written as $\dot{S} = \mathbf{K} = 0$. Equation 28 is effectively the formal solution of the fluid equation of motion (Euler's equation), and thus amounts to a statement of conservation of energy-momentum.

That this is true can be verified by evaluating the stress-energy tensor w_k^j which, since our Lagrangian density is P , is given by :

$$w_k^j \equiv \sum_q p^j \partial_k q - P \delta_k^j \equiv \sum_q [\partial P / \partial (\partial_j q)] \partial_k q - P \delta_k^j \quad (29)$$

which, if the Euler-Lagrange equations $\partial \cdot \mathbf{p} = \partial P / \partial q$ are satisfied, automatically satisfies the equation :

$$\partial_j w_k^j = - (\partial P / \partial x^k)_{q, \dot{q}} = 0 \quad (30)$$

where we have postulated that the pressure function possesses no *explicit* dependence on the space-time coordinates. Using the expressions for the p_s that were given following 25, and making use of 28, we find that 29 becomes :

$$\begin{aligned} w_k^j &= n v^j [\partial_k \mathcal{G} + S \partial_k \mathcal{T} + (\partial_k \xi) \cdot \mathbf{K}] - P \delta_k^j \\ &= n (M + H/c^2) v^j v_k - P \delta_k^j \end{aligned} \quad (31)$$

Making use of the first equation of 26, equation 30 becomes

$$d[(M + H/c^2)\mathbf{v}]/d\tau = n^{-1} \partial P \quad (32)$$

This is just Euler's equation for the fluid, and may be regarded as the determining equation for the molar energy-momentum vector $(M + H/c^2)\mathbf{v}$. But 28 gives an explicit expression for this vector in terms of the four-gradients of the canonical coordinates, so, as previously remarked, 28 constitutes the formal solution of Euler's equation.

It should be noted, incidentally, that when \mathbf{K} becomes constant over any region, the term $(\partial \xi) \cdot \mathbf{K}$ in 28 becomes the gradient of a scalar, and this corresponds to vorticity-free flow⁹ in this region. As previously noted, this physically observable effect is characterized by a degeneracy resulting from the fact that the labelling vector \mathbf{K} is indistinguishable for neighbouring samples of fluid. If the vector \mathbf{K} , and hence ξ , had never been introduced into the formalism, and we had instead introduced the rest-mass energy $M c^2$ simply by replacing \mathcal{G} by $\tilde{\mathcal{G}}$ where $d\tilde{\mathcal{G}}/d\tau \equiv G + M c^2$, we would have arrived at a variational principle implicitly restricted to the case of vorticity-free flow.

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