ENSEMBLE VERSUS TIME AVERAGE PROBABILITIES IN RELATIVISTIC STATISTICAL MECHANICS

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ABSTRACT

Three results will be reported:

(1) Reasons are advanced why discrete probabilities are not Lorentz-invariant. Such probabilities can be obtained as time average probabilities Π_i for state i in frame I. They can also be obtained from an ensemble E_1 of systems which, like the system of interest, are each on average at rest in a certain frame I_0. Such probabilities Q_i transform like the Π_i and ergodicity is then a Lorentz-invariant notion.

(2) If the ensemble is of the usual type (ensemble E_0) whose systems are all at rest in I_0, then the ensemble-based probabilities are Lorentz-invariant. If E_0 is used ergodicity is not a Lorentz-invariant notion.

(3) If entropy is regarded as invariant and entropy maximization is used, the canonical equilibrium probabilities Π_i which one finds contain an extra term which is not usually found. This term will require further discussion.

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1. THE GRAND CANONICAL CONSTRAINTS

Consider within the framework of special relativity, a procedure which is familiar in statistical mechanics, namely the maximization of entropy subject to constraints. When using the grand canonical ensemble these constraints take the form:

$$\sum_i \Pi_{i0} = 1 \tag{1}$$

$$\sum_i \Pi_{i0} E_{i0} = \langle E_0 \rangle_0 \tag{2}$$

$$\sum_i \Pi_{i0} N_{i0} = \langle N_0 \rangle_0 \tag{3}$$

The system states i can change as a result of inter-particle collisions (conceived as point interactions), and collisions with the walls of the container. A state i is assumed to have probability Π_{i0}; E_{i0} and N_{i0} are the energy and particle number appropriate to state i, and $\langle E_0 \rangle_0$ and $\langle N_0 \rangle_0$ their respective mean quantities. The suffix 0 denotes that all quantities are measured in an inertial frame I_0 in which the system appears at rest, and $\langle\ \rangle_0$ means that Π_{i0} has been used in the average.

In keeping with the principle of covariance one must now seek to express these constraints in a general inertial frame I, and must also include the three components of momentum, $\mathbf{P}_i$, in the same way as the energy. Thus:

$$\sum_i \Pi_i = 1 \tag{4}$$

$$\sum_i \Pi_i E_i = \langle E \rangle \tag{5}$$

$$\sum_i \Pi_i \mathbf{P}_i = \langle \mathbf{P} \rangle \tag{6}$$

$$\sum_i \Pi_i N_i = \langle N \rangle \tag{7}$$

These new constraints, 4 to 7, must of course be satisfied in all inertial frames. Since, however, we cannot use an infinite number of them when actually maximizing the entropy, we must therefore find a finite set of constraints which ensure that equations 4 to 7 do in fact hold in all such frames.

To achieve this, it is necessary to know the Lorentz-transformation properties of the quantities involved. First, in keeping with the usual practice, probability Π_i, and particle number, N_i and $\bar{N}$, may be regarded as Lorentz-invariant. It is at once apparent that the equations 4 and 7 will be satisfied in all frames I if and only if they are satisfied in any one frame (such as I_0). The remaining quantities, energy and momentum, are not Lorentz-invariant and must be treated differently. If, as we did in a recent paper[1], one assumes the system to be *inclusive* (i.e. including the energy and momentum due to the stresses in the container) then $\langle \mathbf{P} \rangle$ and $\langle E \rangle$ are the components of a four-vector. Also, in any state i, $\mathbf{P}_i$ and E_i form a four-vector, and thus constraints 5 and 6 may be expressed by

$$\sum_i \Pi_i \{c\mathbf{P}_i, E_i\} = \{c\langle \mathbf{P} \rangle, \langle E \rangle\} \tag{8}$$

The linearity of the Lorentz transformation ensures that if an equation of this form holds in any one inertial frame then it holds in all such frames. Thus equation 8 may conveniently be expressed in the variables of I_0 as

$$\sum_i \Pi_{i0} \{c\mathbf{P}_{i0}, E_{i0}\} = \{\mathbf{0}, \langle E_0 \rangle_0\} \tag{9}$$

remembering that while the mean momentum of the system is zero in I_0, the momentum appropriate to any given state i need not be so.

Suppose, however, that it is desired to avoid taking into account the stresses in the container. One must then use the results applicable to a *confined system*[1]. In this case the mean energy is not the fourth component of a four-vector. Instead it is the enthalpy, $\langle E \rangle + pV$, which, together with the momentum, provides the four components. (Here p is the Lorentz-invariant pressure, and V is the volume, which is subject to the usual Lorentz contraction.) However, in any given state i the system has constant energy, momentum and particle number, and all its particles move freely without collisions which change these quantities. It is thus appropriate to use the transformation for a free system in this case, and to treat energy and momentum as a four-vector.

An immediate difficulty arises. On the LHS of equations 5 and 6 we have four-vectorial quantities which may be transformed to another frame of reference under the Lorentz transformation. On the right are two quantities which are not components of the same four-vector, and can only be transformed to another frame by the introduction of extra terms involving p and V. How can this difficulty be resolved? We arrive here at the notion of a probability Π_i for a discrete state i which is *not* Lorentz-invariant. This is a new suggestion since discrete probabilities are normally considered Lorentz-invariant[2].

2. IMPLICATIONS FOR STATISTICAL MECHANICS

The simple conclusion of section 1 has rather far-reaching consequences. The first of these is that it is in contradiction with any simple-minded relativistic interpretation of ensembles. If a system is on average at rest, statistical mechanics associates with it a representative ensemble of identical systems. At any one time the various available states i of the system are present in this ensemble in proportion to their probabilities Q_{i0} (say). The motion of these systems has never been discussed, as far as we know, it being assumed that they are at rest in the inertial frame I_0 in which the system of interest is at rest. If one assumes this, then one arrives at an invariant probability

$$Q_i = Q_{i0} \text{ (ensemble-based)} \tag{10}$$

For in a general frame I the number of systems in a given state i is the same when the ensemble is viewed from frame I as it is when the ensemble is viewed from frame I_0. This conclusion, based on ensemble-based probabilities, is in contradiction with the result of the preceding section.

A resolution of this paradox is, however, possible. One can consider the system of interest over a long period of time and allot probabilities Π_i to various states i according to the total time for which the system is in this state. These time-based probabilities Π_i are found to transform as one passes from I_0 to I, because of the Lorentz-transformation of the time. We shall put simply

$$\Pi_i = (1 + f_i)\Pi_{i0} \qquad \text{(time-based)} \tag{11}$$

where f_i is a function, to be discussed in section 3. It is by the use of time-based (rather than invariant ensemble-based) probabilities that one may hope to achieve agreement with section 1.

It will be appreciated that the difference between 10 and 11 implies a result about ergodicity. If one confines oneself to just one ensemble as discussed above, and considers a system which is ergodic in its rest frame I_0, then

$$\Pi_{i0} = Q_{i0}, \quad \text{i.e.} \quad \Pi_i = (1 + f_i)\, Q_i$$

It follows that with $f_i \neq 0$ the system is no longer ergodic in I. Thus one can hope to gain agreement with section 1 only by admitting *either* that ergodicity is not a Lorentz-invariant notion, *or* that the idea of an ensemble as a set of systems all strictly at rest in a certain frame of reference (an ensemble E_0) is inapplicable. We favour the latter alternative and regard it as more satisfactory to restrict the motion of the system S_0 of interest, and also the motion of the systems of the ensemble which represents it by the same condition (an ensemble E_1): the systems must be on average at rest in the same frame (I_0). Ergodicity can then become again a Lorentz-invariant notion, namely for ensembles E_1.

The second corollary of these considerations is that the invariance of the entropy cannot be inferred from the Lorentz-invariance of the probabilities Q_i, as has often been done in the past[1,2]. The reason is that it is the probabilities Π_i (not Q_i) which can agree with the considerations of section 1, and they transform as one passes from I_0 to I. The thermodynamic argument for entropy invariance is, of course, not affected by these considerations. One assumes simply that the gradual acceleration of a system from one frame to another is a reversible process which keeps the entropy unchanged.

One may ask for the constraints 4 to 7 to be amended to specify an average enthalpy. However, this does not get over the difficulty that whatever the expressions in these equations, the four-vector for the left-hand sides is $(c\mathbf{P}, E)$, while it is $(c\langle\mathbf{P}\rangle, \langle E\rangle + pV)$ for the right-hand sides.

3. THE TRANSFORMATION OF TIME-BASED PROBABILITIES

The velocity of frame I_0 in frame I is denoted by $\mathbf{w}$. The transformation of equations 5 and 6 to the frame I_0 in which the system is on average at rest will now be carried out. Using 11, one finds

$$\langle E\rangle = \gamma \sum_i \Pi_{i0}(1 + f_i)(E_{i0} + \mathbf{w}\cdot\mathbf{P}_{i0}) \tag{12}$$

$$\langle\mathbf{P}\rangle = \sum_i \Pi_{i0}(1 + f_i)\,[\gamma(\mathbf{P}_{i0\parallel} + (\mathbf{w}/c^2)E_{i0}) + \mathbf{P}_{i0\perp}] \tag{13}$$

where $\mathbf{P}_{i0\parallel}$ and $\mathbf{P}_{i0\perp}$ are respectively the components of $\mathbf{P}_{i0}$ parallel and perpendicular to $\mathbf{w}$. It will be assumed, as a restriction on the system, that in I_0

$$\langle P_0\rangle_0 \equiv \sum_i \mathbf{P}_{i0}\Pi_{i0} = 0 \tag{14}$$

Writing also

$$\langle E_0\rangle_0 \equiv \sum_i E_{i0}\Pi_{i0}$$

one knows that for a confined system[1]

$$\langle E\rangle = \gamma[\langle E_0\rangle_0 + (w^2/c^2)\,pV_0] \tag{15}$$

$$\langle\mathbf{P}\rangle = (\mathbf{w}/c^2)\,\gamma[\langle E_0\rangle_0 + pV_0] \tag{16}$$

The fact that equations 12 and 15 must be identical, and that 13 and 16 must also be identical, yields conditions on the unknown functions f_i. These are from the energy

$$\sum_i f_i(E_{i0} + \mathbf{w}\cdot\mathbf{P}_{i0})\,\Pi_{i0} = (w^2/c^2)\,pV_0 \tag{17}$$

and from the momentum

$$\sum_i f_i(E_{i0}\mathbf{w} + c^2\mathbf{P}_{i0\parallel} + (c^2/\gamma)\mathbf{P}_{i0\perp})\,\Pi_{i0} = pV_0\mathbf{w} \tag{18}$$

From 18 one finds

$$(w^2/c^2)\sum_i f_iE_{i0}\Pi_{i0} + \sum_i f_i\mathbf{w}\cdot\mathbf{P}_{i0}\Pi_{i0} = (w^2/c^2)\,pV_0 \tag{19}$$

and

$$\sum_{i} f_i \mathbf{P}_{i0\perp} \Pi_{i0} = 0 \tag{20}$$

Since 17, 19 and 20 hold for all $\mathbf{w}$, it follows that

$$\sum_{i} f_i E_{i0} \Pi_{i0} = 0 \tag{21}$$

and

$$\sum_{i} f_i \mathbf{P}_{i0} \Pi_{i0} = (\mathbf{w}/c^2)\, pV_0 \tag{22}$$

The equations 21 and 22 are the conditions on the functions f_i. It can be shown[3] that in a one-particle system the time-based probabilities can be transformed so as to make f_i in 11 equal to

$$f_i = \mathbf{w} \cdot \mathbf{P}_{i0}/E_{i0} \tag{23}$$

This theory also yields a pV term such that 23 satisfies 22[4]. Lastly 23 reduces condition 21 to 14 so that the solution 23 does in fact satisfy the general conditions 21 and 22.

4. DISCUSSION

The major difference between the work done here and earlier work is the rejection of the Lorentz-invariance of discrete probabilities. This apparently far-reaching alteration to basic concepts is made easier to understand by noting that the probabilities specified here represent the proportion of time which is spent in a particular state. The transformation factors for the probabilities Π_i arise because the Lorentz transformation of time depends on the velocity in each state i of the system (or particle) under consideration. We considered two possibilities based on the specific f_i expression given by equation 23. (A) If the system velocities depend on the state i, then f_i is different for the various states i. (B) If, however, the velocity is constant (i.e. the velocity is zero in frame I_0), then 23 yields $f_i = 0$, and the probabilities Π_i are Lorentz-invariant.

The difficulty of choosing between these possibilities lies in deciding on what to take as the velocity in I_0 of a system of particles in a given state. One point of view is to say that the mean velocity of all the particles in the system should be considered. Allowing for fluctuations of momentum and energy, this quantity varies between states i, and leads to the non-invariant probabilities described before, and hence to case (A). This approach corresponds to the treatment of a system as *confined*[1], in which the container is disregarded. It leads at once to a statistical description of the pressure in terms of the motion of the particles (e.g. equation 22). The complication of this method is that the time spent by a system in a state i (defined by a set of occupation numbers for particle states) is determined not by the overall system velocity but by the individual velocities of all the particles.

One arrives at case (B) for an inclusive system, defined in ref. 1, if the velocity of the system is taken to be that of the container, fixed at rest in frame I_0. The behaviour of the particles inside the container is discounted, and the momentum $\mathbf{P}_{i0}$ of the system is deduced from its zero velocity in I_0 to be itself zero. Then, by 23, f_i is zero for all states i, and the standard results with Lorentz-invariant probabilities follow at once. The pressure cannot then be calculated by this method, and must be introduced in a normalization factor.

The difference between cases (A) and (B) can most clearly be seen for a one-particle system. Here the probabilities of the system (i.e. the one particle) being in various states can undoubtedly be determined by the time intervals it spends in those states, and these are accordingly altered under a Lorentz transformation. This is case (A). It leads to the standard results for the Lorentz transformation of energy, momentum, pressure etc. of a confined system.

If entropy is regarded as invariant and if an entropy maximization technique is used, a discrepancy occurs: the canonical probability Π_{i0} is found to be

$$\Pi_{i0} = C \exp\left(-[E_{i0} + \tfrac{1}{3}\mathbf{u}_{i0} \cdot \mathbf{P}_{i0}]/kT_0\right) \tag{24}$$

instead of

$$\Pi_{i0} = C \exp\left(-E_{i0}/kT_0\right) \tag{25}$$

as given by conventional theory, and also by approach (B), treating the system as inclusive. Here C is a normalization factor, k is Boltzmann's constant, T_0 is the temperature (measured in I_0), and $\mathbf{u}_{i0}$ is the particle velocity equal to $c^2\mathbf{P}_{i0}/E_{i0}$. There is clearly a discrepancy between equations 24 and 25. This question will be discussed elsewhere.

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