CARNOT CYCLES FOR GENERAL RELATIVISTIC SYSTEMS

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ABSTRACT

A generalization of ordinary Carnot cycles is given for thermodynamic systems with stationary gravitational fields. The two heat reservoirs are assumed to be located at different points in space. In addition to the standard change of thermodynamic quantities the Carnot engine is allowed to change its position during the cycle. A 'generalized Carnot cycle' is then defined by the following process: (1) Connection of the Carnot engine with the first heat reservoir (exchanging heat), (2) Change of position of the Carnot engine from the first to the second heat reservoir, (3) Connection of the Carnot engine with the second heat reservoir (exchanging heat), (4) Change of position of the Carnot engine from the second to the first heat reservoir, after which the cycle repeats. In all changes of position the presence of the gravitational field has to be considered. The special case of an ordinary Carnot cycle is obtained when there is no gravitational field or when the heat reservoirs are located at the same point. Under the assumption that gravitation can be described by general relativity the efficiency of these generalized Carnot cycles is calculated for stationary fields. Thermodynamic equilibrium exists when the efficiency of a generalized Carnot cycle operating between any two parts of the system is zero. For this case we find that $T \cdot \|\xi\|$ is a constant independent of position. As used here T is the ordinary thermodynamic temperature and $\|\xi\|$ denotes the norm field of the Killing vector field ξ, representing the stationarity of the gravitational field. The proof is independent of the field equations of general relativity. Consequently equilibrium consists of a temperature field which depends on the gravitational field. For static fields with spherical symmetry Tolman has proved this relation by using the field equation of general relativity. Our results show that this relation holds quite generally for arbitrary stationary fields.

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1. INTRODUCTION

Classical thermodynamics has been developed with the assumption that either no gravitational fields are present in the system, or that the fields act on the rest-mass of the system or particles only and not on any other kind of internal energy like heat or elastic energy. Yet from special relativity we know that every kind of internal energy has inertia, and from the principle of equivalence of inertial and gravitational mass, it then follows that every kind of internal energy has (passive) gravitational mass.

In order to find the exact thermodynamic relations for systems with gravitational fields one therefore has to take into account explicitly the action of gravitation on internal energy. We assume that gravitation can be described by Einstein's theory of general relativity. The first to work on this problem were Tolman and Ehrenfest[1,2]. By a proposed generalization of the second law of classical thermodynamics to general relativistic systems Tolman[3] derived for two special cases and thermodynamic equilibrium using the field equation of general relativity the so-called Tolman relation $T \cdot \sqrt{g_{00}} =$ const. where g_{00} denotes the time component of the metric tensor of space time. Later Landau and Lifshitz[4] and Balazs[5] derived the same result by generalizing the thermodynamic relation $(\partial S/\partial E) = 1/T$, to general relativistic systems, where S and E denote entropy and energy. In the framework of general relativistic statistical mechanics Ehlers[6] and Tauber and Weinberg[7] derived the Tolman relation for an ideal gas in thermodynamic equilibrium.

The approach to general relativistic thermodynamics given here differs from those used by the above authors. It has no need of a previously defined concept of entropy for general relativistic systems but is rather an operational approach in the sense of Buchdahl[8]. The basic idea, given[9] earlier, is a straightforward generalization of the concept of Carnot cycles. With the help of these generalized Carnot cycles temperature, thermodynamic equilibrium and entropy of general relativistic systems can be defined.

For a weak gravitational field Balazs and Dawson[10] have introduced independently of us this concept of generalized Carnot cycles. Their results are the weak field approximation of the results given here.

2. DEFINITION OF GENERALIZED CARNOT CYCLES

We begin with some plausible suppositions. The considered thermodynamic system with gravitation can always be thought to be divided into arbitrary subsystems, each sufficiently small so that temperature and field quantities can be considered as constant throughout each subsystem but may be different in different subsystems. Let the Carnot engine be a machine which can convert heat into mechanical work to a certain extent, and *vice versa*, and in which the mechanical work can be stored. The engine is supposed to be smaller than each subsystem and its mass to be so small that it does not change the gravitational field when it operates between two subsystems. The subsystems can act like heat reservoirs. A *generalized Carnot cycle* is then defined by the following process:

(1) Connection of the Carnot engine with the first heat reservoir, exchange of heat and conversion of heat into mechanical work stored in the engine.
(2) Change of position of the Carnot engine from the first to the second heat reservoir and adiabatic change of its internal state.
(3) Connection of the Carnot engine with the second heat reservoir, conversion of some of the stored mechanical work into heat and exchange of heat.
(4) Change of position of the Carnot engine from the second to the first heat reservoir and adiabatic change of the internal state back to the state at the beginning of the cycle.

This generalized cycle differs from the ordinary one only by the explicit change of the position of the engine in the gravitational field. By this change heat is transported through the field and according to the equivalence of inertial and gravitational mass this needs mechanical work. The efficiency of the cycle is therefore modified by the field.

A generalized Carnot cycle is reduced to an ordinary one when there is no gravitational field or when the heat reservoirs are located at the same point.

3. EFFICIENCY OF GENERALIZED CARNOT CYCLES

We give a short and abbreviated calculation of the efficiency of a generalized cycle for stationary gravitational fields using the notation of modern differential geometry[11,12]. For a detailed calculation see Ebert and Göbel[13].

Let **X**, **Y** be vectors of the tangent space at an arbitrary point of the Riemannian manifold space-time and let $\langle \mathbf{X}, \mathbf{Y} \rangle$ denote the inner product and $\mathrm{D}_{\mathbf{Y}}\mathbf{X}$ the covariant derivative of **X** in the direction **Y**. From the assumed stationarity of the field there follows the existence of a timelike Killing vector field ξ which satisfies the Killing equation[14].

$$\langle \mathbf{X}, \mathrm{D}_{\mathbf{Y}}\xi \rangle + \langle \mathbf{Y}, \mathrm{D}_{\mathbf{X}}\xi \rangle = 0 \quad \text{for arbitrary } \mathbf{X}, \mathbf{Y} \tag{1}$$

A line in space-time of which all tangent vectors belong to the Killing field ξ is called a *Killing orbit*. Then by a short calculation one gets from equation 1

$$\langle \xi, \xi \rangle = \text{const. on any Killing orbit} \tag{2}$$

and, if τ denotes the unit tangent vector of a geodesic line in space-time,

$$\langle \tau, \xi \rangle = \text{const. on any geodesic line} \tag{3}$$

We now consider the Carnot engine during the generalized cycle. It will move along a world line which first coincides for a certain time interval with the world line of the first heat reservoir, then runs to the world line of the second heat reservoir, coincides with this line for a certain time interval and runs back to the world line of the first heat reservoir. The energy E_p of the Carnot engine at a point p on the world line of the engine measured by an observer for whom the gravitational field is stationary, is given by the inner product of the four-momentum of the engine at p and the unit tangent vector of the observer at the same point. Let the total mass of the engine at p be denoted by m_p and the four-velocity of the engine at p by u_p, then (sign convention: $\langle \mathbf{X}, \mathbf{X} \rangle > 0$ for time-like vectors $\mathbf{X}$; $c = 1$)

$$E_p = m_p \langle u_p, \xi_p \rangle / \|\xi_p\| \tag{4}$$

where $\|\xi_p\| := \{\langle \xi_p, \xi_p \rangle\}^{\frac{1}{2}}$ is called the norm of the Killing vector ξ at p. The mass of the engine is a scalar but it changes its value when the internal energy of the engine is changed, by heat exchange.

Without loss of generality we can accomplish the change of position of the Carnot engine (from the first to the second and from the second to the first heat reservoir) by moving it on geodesic lines (we only have to start the motion with sufficient kinetic energy). Then equation 3 applies for those parts of the world line of the engine which belong to the change of position and equation 2 applies for those parts which represent the connection of the engine with the heat reservoirs (it is assumed that the gravitational field in the whole thermodynamic system is stationary, therefore observers located at subsystems observe also a stationary field).

Let Q_α, Q_β be the absolute values of the heat exchanged with the heat reservoir α (first reservoir) and β (second reservoir) respectively and measured by an observer co-moving with the Carnot engine. Then the mass of the engine changes from the value m in the beginning of the cycle into $m + Q_\alpha$ ($c = 1$) after the first heat exchange, and into $m + Q_\alpha - Q_\beta$ after the second exchange. Taking into account all changes of kinetic and internal energy of the engine and using equations 2, 3 and 4 one finally gets for the mechanical work W gained in one cycle measured by an observer attached to the first heat reservoir α (for detailed calculations see ref. 13)

$$W = Q_\alpha - Q_\beta \frac{\|\xi\|_\beta}{\|\xi\|_\alpha} \tag{5}$$

where $\|\xi\|_\alpha$, $\|\xi\|_\beta$ denote the constant norms of the Killing vectors along the world lines of the reservoirs α and β respectively according to equation 2.

When there is no gravitational field then $\|\xi\|_\alpha = \|\xi\|_\beta = 1$ and equation 5 is reduced to the result of classical thermodynamics. If there is a field but the two heat reservoirs are located at the same point then $\alpha = \beta$ and equation 5 is again reduced to the classical result.

If we choose a coordinate system such that the time coordinate is a parameter on the Killing orbits (which is always possible), then $\langle\xi, \xi\rangle = g_{00}$. Equation 5 then becomes

$$W = Q_\alpha - Q_\beta\sqrt{[g_{00}(\beta)]}/\sqrt{[g_{00}(\alpha)]}$$

where $g_{00}(\alpha)$, $g_{00}(\beta)$ are the time-independent time components of the metric tensor at the heat reservoirs α and β respectively. For the Carnot efficiency η we get from equation 5

$$\eta := W/Q_\alpha = 1 - Q_\beta\|\xi\|_\beta/Q_\alpha\|\xi\|_\alpha$$

or in special coordinates from equation 6

$$\eta = 1 - Q_\beta\sqrt{[g_{00}(\beta)]}/Q_\alpha\sqrt{[g_{00}(\alpha)]} \tag{8}$$

4. DEFINITION OF TEMPERATURE

Before expressing the Carnot efficiency with temperatures instead of heat energies we have to define temperature in a thermodynamic system with gravitational fields. In classical thermodynamics temperature can be defined by using Carnot cycles and the principle of Kelvin[15] which may be stated: *It is impossible to convert an amount of heat completely into work by a cyclic process, without at the same time producing other changes.* By adding Carnot cycles running in opposite directions and using the above principle one gets the well-known result that the ratio of the absolute values Q_1, Q_2 of the exchanged heat energies in one cycle must be a real valued function of the temperatures T_1, T_2 of the heat reservoirs only. Taking into account a functional equation for this function, which results from the possibility of adding two cycles together to form a third one, one defines the absolute or thermodynamic temperature by

$$T_2 := T_1\, Q_2/Q_1 \tag{9}$$

where T_1 has to be fixed by a physical process.

For systems with gravitational fields we define temperature in a completely analogous way. First we make the basic assumption: *Kelvin's principle also holds for systems with stationary gravitational fields.* From here we get for a generalized Carnot cycle the result that the absolute values of the exchanged heat energies Q_α, Q_β in one cycle must be a real valued function f of only the temperatures T_α, T_β of the heat reservoirs *and* of the metric tensor g_{ik} at α and β:

$$Q_\alpha/Q_\beta = \mathrm{f}[T_\alpha, g_{ik}(\alpha); T_\beta, g_{ik}(\beta)] \tag{10}$$

Because of the possibility of adding two cycles together to get a third one a certain functional equation for f has to be fulfilled[13]. Taking into account this equation we define the *thermodynamic temperature of a system with gravitation by*

$$T_\beta := T_\alpha\, Q_\beta/Q_\alpha \tag{11}$$

where T_α has to be fixed by some physical process.

Because Q_α, Q_β are the exchanged heat energies measured by a co-moving observer attached to the engine (or equivalently attached to the heat reservoirs α and β respectively) this temperature can be measured also by an ideal gas thermometer permanently connected with the heat reservoirs α and β respectively. This follows from the fact that a co-moving observer attached to the engine and measuring in proper units finds no difference from classical thermodynamics concerning the temperature as long as he uses the temperature definition 11. The above defined temperature is equal to the *proper temperature* introduced by Tolman[3] in a completely different way.

Using 11 we get for the Carnot efficiency 7 the relation

$$\eta = 1 - \frac{T_\beta \|\xi\|_\beta}{T_\alpha \|\xi\|_\alpha} = \frac{T_\alpha \|\xi\|_\alpha - T_\beta \|\xi\|_\beta}{T_\alpha \|\xi\|_\alpha} \tag{12}$$

or in coordinates

$$\eta = \{T_\alpha \sqrt{[g_{00}(\alpha)]} - T_\beta \sqrt{[g_{00}(\beta)]}\}/T_\alpha \sqrt{[g_{00}(\alpha)]} \tag{13}$$

Because the temperature and the norm of the Killing vector are both $\geqslant 0$ by definition and $g_{00} \geqslant 0$ (see sign convention in connection with equation 4) the relation $\eta \leqslant 1$ holds. As in classical thermodynamics η can never become greater than unity, even for the strongest gravitational fields.

5. THERMODYNAMIC EQUILIBRIUM

In classical thermodynamics two systems are in equilibrium if and only if the Carnot efficiency of a cycle operating between these two systems is zero. As can be seen this holds for systems with gravitational fields also if we use the generalized Carnot efficiency given by equation 12. Therefore equilibrium is characterized by

$$T_\alpha \|\xi\|_\alpha = T_\beta \|\xi\|_\beta \tag{14}$$

The whole system is in equilibrium if equation 14 holds for all cycles operating between any two subsystems, and therefore

$$T \|\xi\| = \text{const.} \tag{15}$$

is the temperature relation in equilibrium. In coordinates equation 15 becomes

$$T\sqrt{g_{00}} = \text{const.} \tag{16}$$

which is the Tolman relation. We see that this relation holds quite generally for arbitrary systems with stationary gravitational fields. In getting equation 15 we did not use the field equations of general relativity, we only used a Riemannian manifold for space-time, the principle of equivalence and special relativity.

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