SOME RECENT RESULTS IN THE THEORY OF FADING MEMORY

BERNARD D. COLEMAN

Mellon Institute and Department of Mathematics, Carnegie–Mellon University, Pittsburgh, Pa 15213, U.S.A.

ABSTRACT

An outline is given of the phenomenological theory of fading memory recently explored by V. J. Mizel and the author. The theory provides a general framework in which one can derive the restrictions which the second law of thermodynamics places on the constitutive equations of materials with memory.

---

1. INTRODUCTION

In theories of the dynamical behaviour of continua, there are several ways of describing the dissipative effects which, in addition to heat conduction, accompany deformation. The oldest way is to employ a viscous stress which depends on the rate of strain, as is done in the theory of Navier–Stokes fluids. In another description of dissipation, one postulates the existence of internal state variables which influence the stress and obey differential equations in which the strain appears. A third approach is to assume that the entire past history of the strain influences the stress in a manner compatible with a general postulate of smoothness or 'principle of fading memory'.

Experience in high-polymer physics shows that the mechanical behaviour of many materials, including polymer melts and solutions, as well as amorphous, crosslinked solids and semi-crystalline plastics, is more easily described within the theory of materials with fading memory than by theories of the viscous-stress type, which do not account for gradual stress-relaxation, or by theories which rest on a finite number of internal state variables and which, therefore, give rise to discrete relaxation spectra when linearized.

Some years ago, Walter Noll and I proposed a systematic procedure for rendering explicit the restrictions which the second law places on constitutive relations[1]. The procedure was easily applied in theories of materials of the viscous-stress type[1, 2] and in theories which employ evolution equations for internal state variables[5]; these applications did not yield results which a physicist would consider surprising and were presented as attempts at clarification, with the emphasis laid upon logical relations. Implementation of the procedure in the theory of materials with memory was a different matter, however, for it there led to conclusions[4] which, although not anticipated by other arguments, have recently been shown to have important bearing on wave propagation[5] and dynamical stability[6, 7]. Here I should like to discuss the restrictions which the second law places on the response functionals of materials with memory. Although it is possible to develop analogous theories for materials with 'permanent memory'†, I emphasize materials which possess 'fading memory' in the sense that configurations experienced in the recent past have a stronger influence on the present values of the stress and free energy than configurations experienced in the distant past.

2. PROCESSES, CONSTITUTIVE ASSUMPTIONS, AND THE SECOND LAW

Let a fixed reference configuration $\mathscr{R}$ be assigned for the body $\mathscr{B}$ under consideration, and identify each of the material points X of $\mathscr{B}$ with the place $\boldsymbol{\xi}$ in space that X occupies when $\mathscr{B}$ has the configuration $\mathscr{R}$. A *thermodynamic process* of $\mathscr{B}$ is a collection of functions of ξ and time compatible with the laws of balance of momentum and energy. For the materials covered by the present theory, each process consists of eight functions: (1) the *motion* $\boldsymbol{\chi}$, with $\mathbf{x} = \boldsymbol{\chi}(\boldsymbol{\xi}, t)$ called the *position* at time t of the material point located at $\boldsymbol{\xi}$ in $\mathscr{R}$, (2) the local absolute *temperature* θ, which is assumed to be positive, (3) the symmetric *stress tensor* $\mathbf{T}$ of Cauchy, (4) the specific *internal energy* ε, per unit mass, (5) the specific *entropy* η, per unit mass, (6) the *heat flux vector* $\mathbf{q}$, (7) the *body force* $\mathbf{b}$, per unit mass (exerted on $\mathscr{B}$ at $\mathbf{x} = \boldsymbol{\chi}(\boldsymbol{\xi}, t)$ by the 'external world', i.e. by other bodies which do not intersect $\mathscr{B}$), and (8) the *rate of heat supply* r (i.e. the radiation energy, per unit mass and unit time, absorbed by $\mathscr{B}$ at $\mathbf{x} = \boldsymbol{\chi}(\boldsymbol{\xi}, t)$, and furnished by the 'external world'). The first six of these functions determine the process, for once $\boldsymbol{\chi}$, θ, $\mathbf{T}$, ε, η, and $\mathbf{q}$ have been specified for all $\boldsymbol{\xi}$ and t, the functions $\mathbf{b}$ and r are determined[1] by the requirement that the process shall obey the laws of balance of momentum and energy, which state that, for each part $\mathscr{P}$ of $\mathscr{B}$ and each time t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \dot{\mathbf{x}}\,\mathrm{d}m = \int_{\mathscr{P}} \mathbf{b}\,\mathrm{d}m + \int_{\partial\mathscr{P}} \mathbf{Tn}\,\mathrm{d}a \tag{2.1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} (\varepsilon + \tfrac{1}{2}\,\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\,\mathrm{d}m = \int_{\mathscr{P}} (\dot{\mathbf{x}}\cdot\mathbf{b} + r)\,\mathrm{d}m + \int_{\partial\mathscr{P}} (\dot{\mathbf{x}}\cdot\mathbf{Tn}) - \mathbf{q}\cdot\mathbf{n})\,\mathrm{d}a \tag{2.2}$$

In these equations $\mathrm{d}m$ is the element of mass in the body, $\partial\mathscr{P}$ is the surface of $\mathscr{P}$ in the configuration at time t, $\mathrm{d}a$ is the element of surface area, $\mathbf{n}$ is the exterior unit normal vector to $\partial\mathscr{P}$, and the superposed dots denote material time-derivatives.

The specific free energy‡ ψ is defined by

$$\psi = \varepsilon - \theta\eta \tag{2.3}$$

† See, for examples, Owen's discussion of the thermodynamics of materials with elastic range[8], Owen and Williams's theory of rate-independent materials[9], and a recent essay[10], in which Owen and I generalize the present treatment.

‡ Also called the 'Helmholtz free energy per unit mass'.

The *deformation gradient* $\mathbf{F}$ is the gradient of $\boldsymbol{\chi}(\boldsymbol{\xi}, t)$ with respect to $\boldsymbol{\xi}$:

$$\mathbf{F} = \mathbf{F}(\boldsymbol{\xi}, t) = \nabla_{\xi}\boldsymbol{\chi}(\boldsymbol{\xi}, t) \tag{2.4}$$

It is assumed that $\mathbf{F}$ is non-singular; hence

$$\det \mathbf{F} \neq 0 \tag{2.5}$$

The Piola–Kirchhoff tensor, $\mathbf{S} = \mathbf{S}(\boldsymbol{\xi}, t)$, is defined by

$$\mathbf{S} = (1/\rho)\,\mathbf{T}\mathbf{F}^{T^{-1}}, \qquad \rho\mathbf{S}\mathbf{F}^{T} = \mathbf{T} \tag{2.6}$$

with ρ the mass density. I denote by $\mathbf{g}$ the spatial gradient of the temperature, i.e. the gradient of θ considered as a function of the present position $\mathbf{x} = \boldsymbol{\chi}(\boldsymbol{\xi}, t)$:

$$\mathbf{g} = \nabla_{\mathbf{x}}\theta(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t) \quad \text{or} \quad \mathbf{F}^{T}\mathbf{g} = \nabla_{\xi}\theta(\boldsymbol{\xi}, t) \tag{2.7}$$

Now, let $\mathbf{F}(\tau)$ and $\theta(\tau)$ be the deformation gradient and temperature at time τ at a fixed material point X. The functions $\mathbf{F}^{t}$ and θ^{t}, defined by

$$\mathbf{F}^{t}(s) = \mathbf{F}(t-s), \quad \theta^{t}(s) = \theta(t-s), \quad 0 \leqslant s < \infty \tag{2.8}$$

are called the *history up to t of the deformation gradient at* X and the *history up to t of the temperature at* X; $\mathbf{F}^{t}$ maps $[0, \infty)$ into the set of non-singular tensors, while θ^{t} maps $[0, \infty)$ into the set of positive numbers.

Each material is characterized by constitutive relations which limit the class of processes possible in a body comprised of the material. In the thermodynamics of materials with memory‡, a *simple material* is one for which the free energy, the stress, the entropy, and the heat flux are determined when the history of the deformation gradient, the history of the temperature, and the present value of the temperature gradient are specified. Thus, at each material point of a simple material there hold equations of the form:

$$\left.\begin{aligned} \psi(t) &= \mathfrak{p}(\mathbf{F}^{t}, \theta^{t}; \mathbf{g}(t)) \\ \eta(t) &= \mathfrak{h}(\mathbf{F}^{t}, \theta^{t}; \mathbf{g}(t)) \\ \mathbf{S}(t) &= \mathfrak{s}(\mathbf{F}^{t}, \theta^{t}; \mathbf{g}(t)) \\ \mathbf{q}(t) &= \mathfrak{q}(\mathbf{F}^{t}, \theta^{t}; \mathbf{g}(t)) \end{aligned}\right\} \tag{2.9}$$

It is assumed that the four functions $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{s}$ and $\mathfrak{q}$ are given at each material point; these functions, called 'response functionals' or 'constitutive functionals' depend, of course, on the choice of the reference configuration§. A process is said to be *admissible* in the simple material if, in addition to obeying the balance laws 2.1 and 2.2, it obeys the constitutive relations 2.9.

If one regards $\mathbf{q}/\theta$ to be a vectorial flux of entropy and r/θ to be a scalar supply of entropy, then it is natural to define the *rate of production of entropy* in a part $\mathscr{P}$ of $\mathscr{B}$ to be

$$\Gamma(\mathscr{P}, t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \eta \,\mathrm{d}m - \left[\int_{\mathscr{P}} \frac{r}{\theta}\,\mathrm{d}m - \int_{\partial\mathscr{P}} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}a\right] \tag{2.10}$$

‡ Cf. Ref. 4.

§ Cf. Noll[11].

The *Clausius–Duhem inequality*[12] is the assertion that

$$\Gamma(\mathscr{P}, t) \geqslant 0 \tag{2.11}$$

In our paper[1] of 1963, Noll and I pointed out that in many branches of continuum physics the second law of thermodynamics can be given a precise mathematical meaning if it is interpreted to be the following principle.

Dissipation Principle. *For every admissible thermodynamic process in a body* $\mathscr{B}$, *the Clausius–Duhem inequality* 2.11 *must hold at all times* t *and in all parts* $\mathscr{P}$ *of* $\mathscr{B}$.

It is clear that this principle implies that response functionals cannot be chosen arbitrarily. In Section 4 I shall list the restrictions which the principle places on $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{s}$ and $\mathfrak{q}$ in 2.9 when these functions obey the postulate of regularity called the 'principle of fading memory'‡. First, however, I should like to outline a recently developed axiomatic approach[13] to the theory of fading memory.

3. ON THE THEORY OF FADING MEMORY

Let us follow a procedure employed in ref. 4, and use Greek majuscules, such as Λ, to denote ordered pairs $(\mathbf{L}, \lambda)$, with $\mathbf{L}$ a tensor and λ a scalar. The definitions

$$\left.\begin{aligned}\alpha\Lambda_1 + \beta\Lambda_2 = \alpha(\mathbf{L}_1,\lambda_1) + \beta(\mathbf{L}_2,\lambda_2) = (\alpha\mathbf{L}_1 + \beta\mathbf{L}_2, \alpha\lambda_1 + \beta\lambda_2),\\ \Lambda_1\cdot\Lambda_2 = \operatorname{tr}(\mathbf{L}_1\mathbf{L}_2^T) + \lambda_1\lambda_2\end{aligned}\right\} \tag{3.1}$$

make the set of all such ordered pairs a 10-dimensional vector space $\mathscr{V}_{(10)}$ with norm

$$|\Lambda| = \sqrt{\Lambda\cdot\Lambda} = \sqrt{\operatorname{tr}(\mathbf{L}\mathbf{L}^T) + \lambda^2} \tag{3.2}$$

The elements of $\mathscr{V}_{(10)}$ of the type

$$\boldsymbol{\Gamma} = (\mathbf{F}, \theta) \tag{3.3}$$

with $\mathbf{F}$ a non-singular tensor and θ a positive number, form a cone $\mathscr{C}$ in $\mathscr{V}_{(10)}$§. At a given material point in a process, the *total history up to t*, i.e. the history up to t of the deformation gradient and temperature, is the function $\boldsymbol{\Gamma}^t = (\mathbf{F}^t, \theta^t)$, mapping $[0, \infty)$ into $\mathscr{C}$:

$$\boldsymbol{\Gamma}^t(s) = (\mathbf{F}^t(s), \theta^t(s)) \quad \text{for } 0 \leqslant s < \infty. \tag{3.4}$$

The ordered pair $\boldsymbol{\Sigma}$, defined by

$$\boldsymbol{\Sigma} = (\mathbf{S}, -\eta) = \left(\frac{1}{\rho}\mathbf{T}\mathbf{F}^{T^{-1}}, -\eta\right) \in \mathscr{V}_{(10)}, \tag{3.5}$$

‡ For earlier studies of the principle of fading memory, see refs. 14–18. Ref. 19 surveys work done up to 1965.

§ A subset $\mathfrak{A}$ of a vector space is called a *cone* if $\mathbf{u} \in \mathfrak{A}$ and $\mathrm{b} > 0$ imply $b \in \mathfrak{A}$

is called the *stress-entropy vector*[4]. If one writes simply ψ for $\psi(t)$, g for g(t), and $\boldsymbol{\Sigma}$ for $\boldsymbol{\Sigma}(t)$, the constitutive equations 2.9 become, in the present notation,

$$\left.\begin{aligned} \psi &= \mathfrak{p}(\boldsymbol{\Gamma}^t; \mathbf{g}) \\ \boldsymbol{\Sigma} &= \mathfrak{S}(\boldsymbol{\Gamma}^t; \mathbf{g}) \\ \mathbf{q} &= \mathfrak{q}(\boldsymbol{\Gamma}^t; \mathbf{g}) \end{aligned}\right\} \tag{3.6}$$

where the response functional $\mathfrak{S}$ has the 'components'

$$\mathfrak{S} = (\mathfrak{s}, -\mathfrak{h}) \tag{3.7}$$

It is frequently possible to prove theorems in a branch of continuum physics without specifying the form of response functionals, but usually one must assume something about their smoothness. For this reason several topologies have been proposed as appropriate for sets of histories[14–18, 4, 8, 9, 10].

Let us suppose that the histories $\boldsymbol{\Lambda}^t$ of interest form a cone $\mathfrak{C}$ in a Banach function-space $\mathfrak{B}$. Certain basic, but usually tacit, assumptions of physical theories place limitations on the choice of the function space $\mathfrak{B}$ and its norm $\|\cdot\|$. I list below three of these requirements‡.

(1) Given an arbitrary history $\boldsymbol{\Gamma}^t$ in the domain D of a constitutive functional§ and a positive number σ, one expects to find in D the history $\boldsymbol{\Gamma}^{t+\sigma}$ for processes in which $\boldsymbol{\Gamma} = (\mathbf{F}, \theta)$ (at some fixed material point) has the history $\boldsymbol{\Gamma}^t$ up to time t and is constant throughout the interval $[t, t+\sigma]$. The history $\boldsymbol{\Gamma}^{t+\sigma}$ in such a process is called the 'static continuation of $\boldsymbol{\Gamma}^t$ by the amount σ'. The static continuation of a history should be well defined even if one identifies the history with the set of functions at zero distance from it in $\mathfrak{B}$.

(2) If the history $\boldsymbol{\Gamma}^t$ of $\boldsymbol{\Gamma}$ up to time t is in the domain D, then one expects to find in D the histories $\boldsymbol{\Gamma}^{t-\sigma}$ of $\boldsymbol{\Gamma}$ up to previous times $t-\sigma, \sigma \geqslant 0$. These earlier histories are called 'σ-sections of $\boldsymbol{\Gamma}^t$'.

(3) Since it should be possible to evaluate response functionals at 'equilibrium states', one expects D to contain constant histories of the form $\boldsymbol{\Gamma}^t(s) \equiv \boldsymbol{\Omega}$, $0 \leqslant s < \infty$.

Victor Mizel and I have found some apparently useful implications of these elementary physical requirements, and I summarize below some of our results[13].

Let μ be an *influence measure*; that is, a non-trivial‖, sigma-finite, positive, regular Borel measure on $[0, \infty)$ and let $\mathscr{s}$ be the set of all μ-measurable functions ϕ mapping $[0, \infty)$ into $[0, \infty)$. Let ν be a function on $\mathscr{s}$ such that for all ϕ (or ϕ_i) in $\mathscr{s}$:

(i) $0 \leqslant \nu(\phi) \leqslant \infty$, and $\nu(\phi) = 0$ if and only if $\phi(s) = 0$ μ-a.e.¶;

(ii) $\nu(\phi_1 + \phi_2) \leqslant \nu(\phi_1) + \nu(\phi_2)$, and $\nu(a\phi) = a\nu(\phi)$ for all numbers $a \geqslant 0$;

‡ Cf. Coleman and Mizel[18, 13].

§ D is here the domain for a fixed value of g.

‖ i.e. not identically zero.

¶ i.e. for all s in $[0, \infty)$ except for a set Z with $\mu(Z) = 0$.

(iii) if $\phi_1(s) \leqslant \phi_2(s)$ μ-a.e., then $\nu(\phi_1) \leqslant \nu(\phi_2)$;
(iv) there is at least one function ψ in $\mathscr{A}$ with $0 < \nu(\psi) < \infty$;
(v) if $\psi, \phi_1, \phi_2, \ldots$ are in $\mathscr{A}$ and if $\phi_n(s) \uparrow \psi(s)$ μ-a.e., then $\nu(\phi_n) \uparrow \nu(\psi)$.

Such a function ν is called a non-trivial function norm, relative to μ, with the sequential Fatou property‡.

Let $\overline{V}$ be the set of μ-measurable functions mapping $[0, \infty)$ into $\mathscr{V}_{(10)}$, and let $\|\cdot\|$ be the function on $\overline{V}$ defined by

$$\|\boldsymbol{\Phi}\| = \nu(|\boldsymbol{\Phi}|) \tag{3.8}$$

for each $\boldsymbol{\Phi}$ in $\overline{V}$. I write V for the set of all $\boldsymbol{\Phi}$ in $\overline{V}$ with $\|\boldsymbol{\Phi}\| < \infty$. Each function $\boldsymbol{\Phi}$ in V is called a *history*, and its independent variable (usually denoted by s) is called the *elapsed time.* The value $\boldsymbol{\Phi}(0)$ of $\boldsymbol{\Phi}$ at $s = 0$ is called the *present value* of $\boldsymbol{\Phi}$, and the *past values* are those for which $0 < s < \infty$. The function space $\mathfrak{V}$ obtained by calling two functions $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ in V the same whenever $\|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\| = 0$ is easily shown to be a Banach space; it is called a *history space* or, at length, a *Banach function space formed from histories with values in* $\mathscr{V}_{(10)}$.

Let C be the class of functions $\boldsymbol{\Phi}$ in V such that $\boldsymbol{\Phi}(s)$ is in $\mathscr{C}$ for all $s \geqslant 0$, and let $\mathfrak{C}$ be the set of equivalence classes obtained by calling the same those elements $\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2$ of C for which $\|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\| = 0$. Clearly, C is a cone in V, and $\mathfrak{C}$ is a cone contained in the Banach function-space $\mathfrak{V}$. Let $\mathfrak{C}$ be the domain D of definition of the response functionals in (3.6)[20]§.

If $\boldsymbol{\Psi}$ is a function on $[0, \infty)$ and σ a positive number, then the *static continuation* of $\boldsymbol{\Psi}$ by the amount σ is the function $\boldsymbol{\Psi}^{(\sigma)}$ on $[0, \infty)$ defined by[16]

$$\boldsymbol{\Psi}^{(\sigma)}(s) = \begin{cases} \boldsymbol{\Psi}(0), & 0 \leqslant s < \sigma \\ \boldsymbol{\Psi}(s-\sigma), & \sigma < s < \infty \end{cases} \tag{3.9}$$

and the *σ-section* of $\boldsymbol{\Psi}$ is the function $\boldsymbol{\Psi}_{(\sigma)}$ on $[0, \infty)$ given by[18]

$$\boldsymbol{\Psi}_{(\sigma)}(s) = \boldsymbol{\Psi}(s + \sigma), \qquad 0 \leqslant s < \infty \tag{3.10}$$

If $\boldsymbol{\Psi}$ is the history up to t of $\boldsymbol{\Gamma} = (\mathbf{F}, \theta)$ (at a fixed material point X in some particular process), then $\boldsymbol{\Psi}_{(\sigma)}$ is the history of $\boldsymbol{\Gamma}$ up to $t - \sigma$, while $\boldsymbol{\Psi}^{(\sigma)}$ gives the history of $\boldsymbol{\Gamma}$ up to $t + \sigma$ assuming that $\boldsymbol{\Gamma}$ is held constant from t to $t + \sigma$. The physical requirements (1) and (2) stated above are made precise by laying down the following two postulates[18, 13].

Postulate 1. *If a given function* $\boldsymbol{\Phi}$ *is in C, then all its static continuations* $\boldsymbol{\Phi}^{(\sigma)}$, $\sigma \geqslant 0$, *are also in C. Furthermore, if* $\boldsymbol{\Phi}$ *and* $\boldsymbol{\Psi}$ *in C are such that* $\|\boldsymbol{\phi} - \boldsymbol{\psi}\| = 0$, *then* $\|\boldsymbol{\Phi}^{(\sigma)} - \boldsymbol{\Psi}^{(\sigma)}\| = 0$ *for all* $\sigma \geqslant 0$.

Postulate 2. *If* $\boldsymbol{\Phi}$ *is in C, then so also are all its σ-sections,* $\boldsymbol{\Phi}_{(\sigma)}$, $\sigma \geqslant 0$.

Employing Postulate 1, one can easily prove the following theorem which shows that the present value $\boldsymbol{\Phi}(0)$ of a history $\boldsymbol{\Phi}$ has a special status, in the

‡ Cf. Luxemburg and Zaanen[21] and the literature quoted by them.

§ When the dependence on g is under discussion, the domain is taken to be the set $\mathfrak{C} \times \mathscr{V}_{(3)}$, which forms a cone in $\mathfrak{V} \oplus \mathscr{V}_{(3)}$.

sense that the norm $\|\boldsymbol{\Phi}\| = \nu(|\boldsymbol{\Phi}|)$ places greater emphasis on $\boldsymbol{\Phi}(0)$ than on any individual past value.

Theorem 1‡. *The influence measure μ must have an atom at $s = 0$ and be absolutely continuous on $(0, \infty)$ with respect to Lebesgue measure.*

Postulates 1 and 2, together, yield

Theorem 2§. *Either $\mu((0, \infty)) = 0$, or Lebesgue measure is absolutely continuous on $(0, \infty)$ with respect to μ.*

Thus the μ-measure of the singleton $\{0\}$ is not zero, and if $\mu((0, \infty))$ is not zero, then an arbitrary subset of $(0, \infty)$ has zero μ-measure if and only if it has zero Lebesgue measure. So as to have a non-trivial theory, *let us assume that $\mu((0, \infty))$ is not zero.* Since the measure μ is employed here only to render precise the expression 'μ-a.e.' in the axioms (i), (iii), (iv) and (v) for ν, Theorems 1 and 2 imply that one can here replace μ with the Borel measure on $[0, \infty)$ that assigns the value 1 to the singleton $\{0\}$ and equals Lebesgue measure when restricted to Borel subsets of $(0, \infty)$.

If $\boldsymbol{\Phi}$ is a function in V, the restriction of $\boldsymbol{\Phi}$ to $(0, \infty)$ is called the *past history* of $\boldsymbol{\Phi}$ and is denoted by $\boldsymbol{\Phi}_r$. Let V_r be the set of all functions $\boldsymbol{\Phi}_r$ obtained by restricting members of V to $(0, \infty)$, and define $\|\cdot\|_r$ on V_r by

$$\|\boldsymbol{\Phi}_r\|_r = \|\boldsymbol{\Phi}\chi_{(0,\infty)}\| \tag{3.11}$$

with $\chi_{(0,\infty)}$ the characteristic function|| of $(0, \infty)$. The *space of past histories* is the function space $\mathfrak{B}_r$ obtained by calling the same those elements $\boldsymbol{\Phi}_r$, $\boldsymbol{\Psi}_r$ of V_r for which $\|\boldsymbol{\Phi}_r - \boldsymbol{\Psi}_r\|_r = 0$. It is easily verified that $\|\cdot\|_r$ is a norm on $\mathfrak{B}_r$ and that $\mathfrak{B}_r$ is a Banach space. I write C_r for the set of functions in V_r with values in $\mathscr{C}$ and $\mathfrak{C}_r$ for the corresponding cone in $\mathfrak{B}_r$.

An immediate consequence of Theorems 1 and 2 is

Theorem 3 ¶. *$\mathfrak{B} = \mathscr{V}_{(10)} \oplus \mathfrak{B}_r$, and the norm $\|\cdot\|$ on $\mathfrak{B}$ is equivalent to the norm $\|\cdot\|'$ defined by*

$$\|\boldsymbol{\Phi}\|' = |\boldsymbol{\Phi}(0)| + \|\boldsymbol{\Phi}_r\|_r \tag{3.12}$$

Here $|\cdot|$ is the original norm 3.2 on $\mathscr{V}_{(10)}$, $\|\cdot\|$ is the norm on $\mathfrak{B}$ defined in 3.8, and $\|\cdot\|_r$ is the norm on $\mathfrak{B}_r$ defined in 3.11. The equivalence of $\|\cdot\|'$ and $\|\cdot\|$ means that there exist two positive numbers c_1 and c_2 such that

$$c_1\|\boldsymbol{\Phi}\| \leqslant \|\boldsymbol{\Phi}\|' \leqslant c_2\|\boldsymbol{\Phi}\|$$

for all $\boldsymbol{\Phi}$ in $\mathfrak{B}$. It follows from Theorem 3 that, even after the functions in V are grouped together to form the equivalence classes comprising $\mathfrak{B}$, each history $\boldsymbol{\Phi}$ has a well-defined present value $\boldsymbol{\Phi}(0)$.

If $\boldsymbol{\Omega}$ is a vector, $\boldsymbol{\Omega}$‡ denotes the constant function on $[0, \infty)$ with value $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega}^\ddagger(s) = \boldsymbol{\Omega}, \qquad 0 \leqslant s < \infty \tag{3.13}$$

‡ Ref. 13, Thm 2.1.

§ Ref. 13, Thm 2.2.

|| i.e. $\chi_{(0,\infty)}$ has domain $[0, \infty)$ and is such that $\chi_{(0,\infty)}(s) = 1$ for $s \in (0, \infty)$, while $\chi_{(0,\infty)}(0) = 0$.

¶ Ref. 13, Thm 3.1.

The following postulate embodies the third of the physical requirements listed above.

Postulate 3. *The space* $\mathfrak{V}$ *contains non-trivial constant functions. That is, for each vector* $\boldsymbol{\Omega}$ *in* $\mathscr{C}$, *the function* $\boldsymbol{\Omega}^{\dagger}$ *is in* C.

It follows from this assumption that given any functional $\mathfrak{f}$ on $\mathfrak{C}$, one can define a function $\mathfrak{f}^{\circ}$ on $\mathscr{C}$ by the formula

$$\mathfrak{f}^{\circ}(\boldsymbol{\Omega}) = \mathfrak{f}(\boldsymbol{\Omega}^{\dagger}) \qquad \text{for all} \quad \boldsymbol{\Omega} \in \mathscr{C} \tag{3.14}$$

$\mathfrak{f}^{\circ}$ is called the *equilibrium response function corresponding to* $\mathfrak{f}$.[4] If $\mathfrak{f}$ is a continuous functional on $\mathfrak{C}$, then $\mathfrak{f}^{\circ}$ is continuous on $\mathscr{C}$.

The norm $\|\cdot\|$ on $\mathfrak{V}$ is said to have the *relaxation property*‡, if, for each function $\boldsymbol{\Phi}$ in V,

$$\lim_{\sigma \to \infty} \|\boldsymbol{\Phi}^{(\sigma)} - \boldsymbol{\Phi}(0)^{\dagger}\| = 0 \tag{3.15}$$

where, in accord with 3.13, $\boldsymbol{\Phi}(0)^{\dagger}$ is the constant function on $[0, \infty)$ with value $\boldsymbol{\Phi}(0)$. Clearly, $\|\cdot\|$ has the relaxation property if and only if 3.15 holds for each $\boldsymbol{\Phi}$ in C. Hence the assumption of the relaxation property is equivalent to the assertion that every continuous functional $\mathfrak{f}$ on $\mathfrak{C}$ obeys the relation

$$\lim_{\sigma \to \infty} \mathfrak{f}(\boldsymbol{\Phi}^{(\sigma)}) = \mathfrak{f}(\boldsymbol{\Phi}(0)^{\dagger}) = \mathfrak{f}^{\circ}(\boldsymbol{\Phi}(0)) \tag{3.16}$$

for each $\boldsymbol{\Phi}$ in $\mathfrak{C}$; that is, in the limit of large σ, the response $\mathfrak{f}(\boldsymbol{\Phi}^{(\sigma)})$ to the static continuation $\boldsymbol{\Phi}^{(\sigma)}$ of an arbitrary history $\boldsymbol{\Phi}$ depends only on the present value of $\boldsymbol{\Phi}$ and is given by the equilibrium response function defined in 3.14.

Postulate 4. *The norm* $\|\cdot\|$ *has the relaxation property.*

Postulates 1 to 4 yield

Theorem 4. *Let* $\Lambda_1(\cdot)$ *and* $\Lambda_2(\cdot)$ *be functions mapping* $(-\infty, \infty)$ *into* $\mathscr{V}_{(10)}$ *such that, for each t, the histories* Λ_1^t *and* Λ_2^t *are in V. If* $\lim_{t \to \infty} |\Lambda_1(t) - \Lambda_2(t)| = 0$, *then* $\lim_{t \to \infty} \|\Lambda_1^t - \Lambda_2^t\| = 0$.

A function $\boldsymbol{\Phi}$ in C is called a *tame history* if:

(α) $\boldsymbol{\Phi}$ is differentiable in the classical sense at $s = 0$; that is

$$\dot{\boldsymbol{\Phi}}(0) \overset{\text{def}}{=} -\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\Phi}(s)\bigg|_{s=0} = \lim_{s \to 0^+} \frac{\boldsymbol{\Phi}(0) - \boldsymbol{\Phi}(s)}{s} \tag{3.17}$$

exists.

(β) $\boldsymbol{\Phi}_r$, the past history of $\boldsymbol{\Phi}$, is an absolutely continuous function on $(0, \infty)$.

(γ) $\mathfrak{V}$ contains an element $\dot{\boldsymbol{\Phi}}$, called the *time-derivative* of $\boldsymbol{\Phi}$, which obeys the equation

$$\dot{\boldsymbol{\Phi}}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\Phi}(s), \qquad \mu\text{-a.e.} \tag{3.18}$$

For technical reasons, one assumes

‡ Ref. 18; see also refs. 16, 17, 4, 13 and 20.

Postulate 5. *Tame histories with time-derivatives of compact support are dense in $\mathfrak{C}$. That is, given any $\boldsymbol{\Psi}$ in $\mathfrak{C}$ and any $\delta > 0$, there exists a tame history $\boldsymbol{\Phi}$ in $\mathfrak{C}$ such that $\|\boldsymbol{\Psi} - \boldsymbol{\Phi}\| < \delta$ and $\dot{\boldsymbol{\Phi}}(s) = \mathbf{0}$ for all s outside a closed bounded set in* $[0, \infty)$.

It follows from Postulate 5 that $\mathfrak{B}$ is separable, that continuous functions of compact support are dense in $\mathfrak{B}$, and that $\mathfrak{B}$ has the following *dominated-convergence property*‡ familiar in the theory of Lebesgue spaces: *If* $\boldsymbol{\Psi}$ belongs to $\mathfrak{B}$ and if $\boldsymbol{\Phi}_j$ is a sequence of elements of $\mathfrak{B}$ with $|\boldsymbol{\Phi}_j(s)| \leqslant |\boldsymbol{\Psi}(s)|$, μ-a.e., such that $\boldsymbol{\Phi}_j(s) \to \mathbf{0}$, μ-a.e., *then* $\|\boldsymbol{\Phi}_j\| \to 0$.

Let $\mathfrak{f}$ be a continuous function mapping $\mathfrak{C}$ into a metric space. It follows from Theorem 3 that $\mathfrak{f}$ can be regarded equally well as a function of ordered pairs $(\boldsymbol{\Phi}(0), \boldsymbol{\Phi}_r)$ with $\boldsymbol{\Phi}(0)$ in $\mathscr{C}$ and $\boldsymbol{\Phi}_r$ in $\mathfrak{C}_r$, i.e.

$$\mathfrak{f}(\boldsymbol{\Phi}) = \mathfrak{f}(\boldsymbol{\Phi}(0); \boldsymbol{\Phi}_r) \tag{3.19}$$

and the continuity of $\mathfrak{f}$ over $\mathfrak{C}$ implies that $\mathfrak{f}(\boldsymbol{\Phi}(0); \boldsymbol{\Phi}_r)$ is jointly continuous in the two variables, $\boldsymbol{\Phi}(0)$ in $\mathscr{C}$, and $\boldsymbol{\Phi}_r$ in $\mathfrak{C}_r$. Now, $\mathfrak{f}$ can be used to define a functional transformation mapping functions $\Lambda(\cdot)$ on $(-\infty, \infty)$ into functions $\phi(\cdot)$ on $(-\infty, \infty)$, by setting

$$\phi(t) = \mathfrak{f}(\Lambda^t) \tag{3.20}$$

for each $t \in (-\infty, \infty)$, where $\Lambda^t(s) = \Lambda(t - s)$, $s \geqslant 0$. Employing Postulates 1, 2, 3 and 5, one can prove that the functional transformation, $\Lambda(\cdot) \rightsquigarrow \phi(\cdot)$, preserves regularity in the following sense.

Theorem 6§. *Let $\mathfrak{f}$ be a continuous function on $\mathfrak{C}$ with values in a metric space, and suppose that $\Lambda(\cdot)$ is a function on $(-\infty, \infty)$ with Λ^t in $\mathfrak{C}$ for each t. If $\Lambda(\cdot)$ is a regulated function, i.e. a function for which the limits* $\lim_{\tau \to t^+} \Lambda(\tau)$ *and* $\lim_{\tau \to t^-} \Lambda(\tau)$ *exist for each t, then $\phi(\cdot)$, given by* 3.20, *is also a regulated function. Furthermore, $\phi(\cdot)$ can suffer discontinuities at only those times t_i at which $\Lambda(\cdot)$ is discontinuous; at all other times $\phi(\cdot)$ is continuous.*

(To obtain this result one first shows that the mapping $t \rightsquigarrow \Lambda_r^t \in \mathfrak{C}_r$ is continuous, for all t, even for those at which $\Lambda(\cdot)$ experiences a discontinuity.)

Let $\mathfrak{U}$ be a cone in a Banach space $\mathfrak{B}$, and let $\mathfrak{D}$ be the subspace of $\mathfrak{B}$ spanned by $\mathfrak{U}$. A real-valued function $\mathfrak{g}$ defined on $\mathfrak{U}$ is said to be *continuously Fréchet-differentiable on* $\mathfrak{U}$|| if, for each ϕ in $\mathfrak{U}$ and every ξ in $\mathfrak{B}$ with $\phi + \xi$ in $\mathfrak{U}$,

$$\mathfrak{g}(\phi + \xi) = \mathfrak{g}(\phi) + d\mathfrak{g}(\phi|\xi) + o(\|\xi\|) \tag{3.21}$$

where $d\mathfrak{g}(\cdot|\cdot)$ is defined and continuous on $\mathfrak{U} \times \mathfrak{D}$ and is such that $d\mathfrak{g}(\phi|\xi)$ is a linear function of ξ for each ϕ. The linear functional $d\mathfrak{g}(\phi|\cdot)$ is called the *Fréchet derivative of* $\mathfrak{g}$ *at* ϕ.

An argument given in ref. 20 here yields

‡ For theorems of this type, see Luxemburg and Zaanen[21], Thm 2.2, and Luxemburg[22], Thm 46.2. See also ref. 13, Remarks 3.1 and 3.2.

§ Ref. 13, Remark 3.3; see also ref. 18, Remark 5.1.

|| Of course the definition can be employed for other types of subsets of $\mathfrak{B}$, such as open subsets.

Theorem 7‡. (Chain Rule.) *If* $\mathfrak{g}$ *is a real-valued continuously Fréchet-differentiable function on* $\mathfrak{C}$, *then, for each tame history* $\boldsymbol{\Phi}$ *in* $\mathfrak{C}$, *the derivative*

$$\dot{\mathfrak{g}} \overset{\text{def}}{=} \lim_{\sigma \to 0^+} \frac{\mathfrak{g}(\boldsymbol{\Phi}) - \mathfrak{g}(\boldsymbol{\Phi}_{(\sigma)})}{\sigma} \tag{3.22}$$

exists and is given by

$$\dot{\mathfrak{g}} = d\mathfrak{g}(\boldsymbol{\Phi}|\dot{\boldsymbol{\Phi}}) \tag{3.23}$$

with $\dot{\boldsymbol{\Phi}}$ *the time derivative defined in* 3.18.

Suppose $\mathfrak{g}$ is continuously Fréchet-differentiable on $\mathfrak{C}$, and recall that $\mathfrak{g}(\boldsymbol{\Phi})$ can be written

$$\mathfrak{g}(\boldsymbol{\Phi}) = \mathfrak{g}(\boldsymbol{\Phi}(0);\boldsymbol{\Phi}_r) \tag{3.24}$$

where $\boldsymbol{\Phi}(0)$, in $\mathscr{C}$, is the present value of $\boldsymbol{\Phi}$, and $\boldsymbol{\Phi}_r$, in $\mathfrak{C}_r$, is the restriction of $\boldsymbol{\phi}$ to $(0, \infty)$. The assumed differentiability of $\mathfrak{g}$ on $\mathfrak{C}$ implies the existence, for each $\boldsymbol{\Phi}$, of the *instantaneous derivative*[4] $D\mathfrak{g}(\boldsymbol{\Phi})$ and the *past-history derivative* $\delta\mathfrak{g}(\boldsymbol{\Phi}|\cdot)$, which are determined by the equations

$$\mathfrak{g}(\boldsymbol{\Phi}(0) + \boldsymbol{\Omega}; \boldsymbol{\Phi}_r) = \mathfrak{g}(\boldsymbol{\Phi}(0);\boldsymbol{\Phi}_r) + D\mathfrak{g}(\boldsymbol{\Phi})\cdot\boldsymbol{\Omega} + o(|\boldsymbol{\Omega}|) \tag{3.25}$$

and

$$\mathfrak{g}(\boldsymbol{\Phi}(0);\boldsymbol{\Phi}_r + \boldsymbol{\Psi}_r) = \mathfrak{g}(\boldsymbol{\Phi}(0);\boldsymbol{\Phi}_r) + \delta\mathfrak{g}(\boldsymbol{\Phi}|\boldsymbol{\Psi}_r) + o(\|\boldsymbol{\Psi}_r\|_r); \tag{3.26}$$

3.25 holds for all $\boldsymbol{\Omega}$ in $\mathscr{V}_{(10)}$ with $\boldsymbol{\Phi}(0) + \boldsymbol{\Omega}$ in $\mathscr{C}$, while 3.26 holds for all $\boldsymbol{\Psi}_r$ in $\mathfrak{B}_r$ with $\boldsymbol{\Phi}_r + \boldsymbol{\Psi}_r$ in $\mathfrak{C}_r$. For each $\boldsymbol{\Phi}$ in $\mathfrak{C}$, the value $D\mathfrak{g}(\boldsymbol{\Phi})$ of $D\mathfrak{g}$ is a vector in $\mathscr{V}_{(10)}$, and $\delta\mathfrak{g}(\boldsymbol{\Phi}|\cdot)$ is a linear function on $\mathfrak{B}_r$. The functionals $D\mathfrak{g}$ and $\delta\mathfrak{g}$ determine $d\mathfrak{g}$ through the relation

$$d\mathfrak{g}(\boldsymbol{\Phi}|\boldsymbol{\Psi}) = D\mathfrak{g}(\boldsymbol{\Phi})\cdot\boldsymbol{\Psi}(0) + \delta\mathfrak{g}(\boldsymbol{\Phi}|\boldsymbol{\Psi}_r) \tag{3.27}$$

and one can write the chain rule 3.23 in the form

$$\dot{\mathfrak{g}} = D\mathfrak{g}(\boldsymbol{\Phi})\cdot\dot{\boldsymbol{\Phi}}(0) + \delta\mathfrak{g}(\boldsymbol{\Phi}|\dot{\boldsymbol{\Phi}}_r) \tag{3.28}$$

with $\dot{\boldsymbol{\Phi}}(0)$ the present value, and $\dot{\boldsymbol{\Phi}}_r$ the past history, of $\dot{\boldsymbol{\Phi}}$.

There is now assembled here apparatus sufficient for a precise statement of the principle of fading memory as used in the thermodynamics of simple materials.

Postulate of Smoothness for Response Functionals. *For each simple material there exists a history space* $\mathfrak{B}$, *formed as described above, such that*

(1) $\mathfrak{C}$, *the cone in* $\mathfrak{B}$ *corresponding to functions mapping* $[0, \infty)$ *into* $\mathscr{C}$, *obeys Postulates* 1–5;

(2) *the functionals* $\mathfrak{p}$, $\mathfrak{S}$ *and* $\mathfrak{q}$ *in* 3.6 *are defined and continuous on* $\mathfrak{C} \times \mathscr{V}_{(3)}$§;

(3) *the functional* $\mathfrak{p}$ *is continuously Fréchet-differentiable on* $\mathfrak{C} \times \mathscr{V}_{(3)}$‖.

‡ Ref. 20, Remark 1 and Appendix II; see also refs. 4 and 23. The proof given in ref. 20 does not require Postulate 4.

§ In 3.6, $\boldsymbol{\Gamma}^t \in \mathfrak{C}$, while $\mathbf{g} \in \mathscr{V}_{(3)}$.

‖ $\mathfrak{C} \times \mathscr{V}_{(3)}$ is here considered a cone in $\mathfrak{B} \oplus \mathscr{V}_{(3)}$.

4. CONSEQUENCES OF THE SECOND LAW

It is easily shown that under appropriate assumptions of regularity for the dependence of $\boldsymbol{\chi}$ and θ upon ξ and t, it follows from the balance laws 2.1 and 2.2 that the Clausius–Duhem inequality 2.11 can be written in the form[4]

$$\dot{\psi} \leqslant \boldsymbol{\Sigma}\cdot\dot{\boldsymbol{\Gamma}} - \frac{1}{\rho\theta}\mathbf{q}\cdot\mathbf{g} \tag{4.1}$$

Working with this local form of the inequality one can prove the following theorem which gives the restrictions which the second law places on the response functionals $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{s}$ and $\mathfrak{q}$ in 2.9 [or, equivalently, $\mathfrak{p}$, $\mathfrak{S}$ and $\mathfrak{q}$ in 3.6].

Theorem 8‡. *It follows from the Dissipation Principle and the Postulate of Smoothness for Response Functionals that*

(i) *the functionals* $\mathfrak{p}$ *and* $\mathfrak{S}$ *are independent of* $\mathbf{g}$; i.e.

$$\psi(t) = \mathfrak{p}(\boldsymbol{\Gamma}^t), \qquad \boldsymbol{\Sigma}(t) = \mathfrak{S}(\boldsymbol{\Gamma}^t) \tag{4.2}$$

whenever $\boldsymbol{\Gamma}^t$ *is in* $\mathfrak{C}$;

(ii) *the functional* $\mathfrak{S}$ *is determined by the functional* $\mathfrak{p}$ *through the 'generalized stress relation'*

$$\mathfrak{S} = \mathrm{D}\mathfrak{p}, \tag{4.3}$$

i.e.

$$\boldsymbol{\Sigma}(t) = \mathrm{D}\mathfrak{p}(\boldsymbol{\Gamma}^t), \tag{4.4}$$

whenever $\boldsymbol{\Gamma}^t$ *is in* $\mathfrak{C}$;

(iii) *for each tame history* $\boldsymbol{\Gamma}^t$ *in* $\mathfrak{C}$

$$\delta\mathfrak{p}(\boldsymbol{\Gamma}^t|\dot{\boldsymbol{\Gamma}}^t) \leqslant 0 \tag{4.5}$$

and

$$\mathfrak{q}(\boldsymbol{\Gamma}^t;\mathbf{g})\cdot\mathbf{g} \leqslant -\rho\theta\delta\mathfrak{p}(\boldsymbol{\Gamma}^t;\mathbf{g}) \tag{4.6}$$

Furthermore, (i), (ii) *and* (iii), *when taken together, give not only a necessary, but also a sufficient condition on* $\mathfrak{p}$, $\mathfrak{S}$ *and* $\mathfrak{q}$ *for* (4.1) *to hold for all* $\mathbf{g}$ *in* $\mathscr{V}_{(10)}$ *and all tame histories* $\boldsymbol{\Gamma}^t$ *in* $\mathfrak{C}$.

When I gave this theorem in my essay[4] of 1964, I proved it using a form of the principle of fading memory less general than that described here§. In the present terminology one can say that I employed the Postulate of Smoothness stated at the end of Section 3, but used for $\mathfrak{B}$ a special type of history space, namely a Hilbert space $\mathfrak{H}$ formed from functions $\boldsymbol{\Phi}$, mapping $[0, \infty)$ in $\mathscr{V}_{(10)}$, for which

$$\|\boldsymbol{\Phi}\|^2 \stackrel{\text{def}}{=} |\boldsymbol{\Phi}(0)|^2 + \int_0^\infty |\boldsymbol{\Phi}(s)|^2\, k(s)\,\mathrm{d}s \tag{4.7}$$

‡ Ref. 4, Thm 1, p 19; see also ref. 20, Thm 1.

§ The form of the principle used in ref. 4 was drawn from earlier work done in collaboration with Walter Noll[14, 15, 16].

is finite; $k(s)$ was a fixed, positive, monotone-decreasing function, assumed to be summable on $(0, \infty)$, and called the 'influence function'‡. Later, Victor Mizel and I[20] observed that Theorem 8 remains valid in the present more general theory.

The conclusions (i), (ii) and (iii) of Theorem 8 have some interesting consequences which I list below.

Theorem 9§. *Of all total histories ending with a given value of $\boldsymbol{\Gamma} = (\mathbf{F}, \theta)$, that corresponding to constant values of $\boldsymbol{\Gamma}$ for all times has the least free energy; i.e. for each $\boldsymbol{\Gamma}^t$ in $\mathfrak{C}$*

$$\mathfrak{p}^\circ(\boldsymbol{\Gamma}^t(0)) \leqslant \mathfrak{p}(\boldsymbol{\Gamma}^t) \qquad (4.8)\|$$

Theorem 10¶*If $\boldsymbol{\Gamma}$ is a vector in $\mathscr{C}$, and if $\boldsymbol{\Gamma}^\dagger$ is the constant function defined by $\boldsymbol{\Gamma}^\dagger(s) \equiv \boldsymbol{\Gamma}$, then for all functions $\boldsymbol{\Phi}_r$ in $\mathfrak{V}_r$,*

$$\delta p(\boldsymbol{\Gamma}^\dagger | \boldsymbol{\Phi}_r) = 0 \qquad (4.9)$$

Theorem 11*. *The equilibrium response functions corresponding to $\mathfrak{p}$ and $\mathfrak{S}$ obey the classical relation*

$$\mathfrak{S}^\circ(\boldsymbol{\Gamma}) = \nabla \mathfrak{p}^\circ(\boldsymbol{\Gamma}) \quad \text{for all } \boldsymbol{\Gamma} \text{ in } \mathscr{C} \qquad (4.10)$$

where $\nabla \mathfrak{p}^\circ$ is the ordinary gradient of $\mathfrak{p}^\circ$.

Although the proof of Theorem 8 does not employ Postulate 4, the proofs of Theorems 9, 10 and 11 do**.

ACKNOWLEDGEMENT

I am grateful to the U.S. National Science Foundation for making possible my attendance at the Conference and to the U.S. Air Force Office of Scientific Research for supporting parts of the research reported here.

REFERENCES

1 B. D. Coleman and W. Noll, *Arch. Ration. Mech. Anal.* **13**, 167 (1963).
2 B. D. Coleman and V. J. Mizel, *J. Chem. Phys.* **40**, 1116 (1964).
3 B. D. Coleman and M. E. Gurtin, *J. Chem. Phys.* **47**, 597 (1967).
4 B. D. Coleman, *Arch. Ration. Mech. Anal.* **17**, 1, 230 (1964).
5 B. D. Coleman and M. E. Gurtin, *Arch. Ration. Mech. Anal.* **19**, 266, 317 (1965).
6 B. D. Coleman and V. J. Mizel, *Arch. Ration. Mech. Anal.* **29**, 105 (1968); **30**, 173 (1968).
7 B. D. Coleman and E. H. Dill, *Arch. Ration. Mech. Anal.* **30**, 197 (1968).
8 D. R. Owen, *Arch. Ration. Mech. Anal.* **31**, 91 (1968).

---

‡ Cf. refs. 14–16. No further assumptions on k are needed for the main theorems of ref. 4. See ref. 18 for an axiomatic approach to history spaces obeying 4.7.

§ Ref. 4, Thm 3, p 26; see also ref. 20, Thm 2.

|| See equation 3.14 for the definition of $\mathfrak{p}^\circ$, the equilibrium response function corresponding to $\mathfrak{p}$.

¶ Ref. 4, Corollary to Thm 3, p 26; ref. 18, Thm 3.

* Ref. 4, Remark 11, p 27; see also ref. 20, Thm 4.

** For further discussion of this point, see ref. 10 and the papers referred to therein.

[9] D. R. Owen and W. O. Williams, *Arch. Ration. Mech. Anal.* **33**, 288 (1969).
[10] B. D. Coleman and D. R. Owen, *Arch. Ration. Mech. Anal.* **36**, 245 (1970).
[11] W. Noll, *Arch. Ration. Mech. Anal.* **2**, 197 (1958); **27**, 1 (1967).
[12] C. Truesdell and R. A. Toupin, *The Classical Field Theories*, in the *Encyclopedia of Physics*, Vol. III/1, Springer: Heidelberg (1960).
[13] B. D. Coleman and V. J. Mizel, *Arch. Ration. Mech. Anal.* **29**, 18 (1968).
[14] B. D. Coleman and W. Noll, *Arch. Ration. Mech. Anal.* **6**, 355 (1960).
[15] B. D. Coleman and W. Noll, *Rev. Mod. Phys.* **33**, 239 (1961); errata **36**, 1103 (1964).
[16] B. D. Coleman and W. Noll, *Proc. Internat. Symp. Second-order Effects, Haifa*, 530 (1962).
[17] C.-C. Wang, *Arch. Ration. Mech. Anal.* **18**, 117 (1965).
[18] B. D. Coleman and V. J. Mizel, *Arch. Ration. Mech. Anal.* **23**, 87 (1966).
[19] C. Truesdell and W. Noll, *The Non-linear Field Theories of Mechanics*, in the *Encyclopedia of Physics*, Vol. III/3, Springer: Heidelberg (1965).
[20] B. D. Coleman and V. J. Mizel, *Arch. Ration. Mech. Anal.* **27**, 255 (1967).
[21] W. A. J. Luxemburg and A. C. Zaanen, *Math. Annalen*, **149**, 150 (1963); **162**, 337 (1966).
[22] W. A. J. Luxemburg, *Indag. Math.* **27**, 229 (1965).
[23] V. J. Mizel and C.-C. Wang, *Arch. Ration. Mech. Anal.* **23**, 124 (1966).