RESOLUTION OF THE HIERARCHY OF MANY-BODY DISTRIBUTION FUNCTIONS

S. Fujita

Department of Physics and Astronomy, State University of New York at Buffalo, Buffalo, New York, U.S.A.

ABSTRACT

The rigorous evolution equation for the one-body distribution function $f(t)$ describing an imperfect gas is not closed according to the investigations of Bogoliubov, Born, Green, Kirkwood, Yvon and others (B–B–G–K–Y hierarchy). The analysis of this equation is made in terms of connected diagrams. It is shown that in the bulk limit this equation can rigorously be transformed into an equation which is *closed* with respect to $f(t)$ and which contains a given initial correlation. The latter equation forms a basis of discussing various problems related to irreversibility and transport phenomena.

1. INTRODUCTION

It has been known for many years that the *Boltzmann equation* provides a good description of transport phenomena for a dilute gas of particles interacting with short-range forces. While this equation is a *closed* equation with respect to the one-body distribution function $f(\mathbf{rp}, t)$, the rigorous evolution equation for f is *not* closed according to the investigations by Bogoliubov, Born, Green, Kirkwood, Yvon, and others[1,2]. Recently much effort has been made to derive and generalize the former equation from the latter by introducing approximations. Unfortunately, most of the approximations previously proposed by various authors seem to be motivated by mathematical tractability rather than physical reasoning. To this category of approximations belongs Bogoliubov's f-functional dependence assumption of many-body distribution functions[1], truncation of the hierarchy through random-phase arguments[3], the factorizability of the initial many-body densities into one-body densities[1,4], and others. Although these assumptions were employed to obtain highly useful results in certain instances, since they were introduced at the beginning of the theories, the validities of the theories and their results are often not clear.

In 1955, Van Hove developed a different approach to the problem[5]. By introducing an infinite-order time-dependent perturbation theory, he attempted to determine the structure of a *collision operator* which describes a general interaction process. Since then, this search for a collision operator has been pursued by many, including Kohn and Luttinger[6], Prigogine, and others[7–9]. In particular, Balescu[8] was successful in determining the collision term for a plasma, which takes account of the dynamic Coulomb screening. This collision term is now known as Balescu–Lenard's term[8,10]. A notable advantage of this approach is that if successful it could clarify

the validities of the above-mentioned various assumptions in addition to its own merit of providing a computational method for a transport coefficient.

In 1957 Kubo published a classic work in the theory of transport phenomena[11]. By solving the Liouville equation to first order in the external electric field, he formulated a rigorous closed expression for the electric conductivity in microscopic terms without any guesswork in regard to the collision contribution, which was the crux of the derivation of the Boltzmann equation. This formula is, today, known as the *current correlation function formula*[12]. Concurrently, much effort has been made to formulate other transport coefficients, such as the viscosity coefficient, which are related to thermodynamic perturbations rather than electromagnetic forces[13]. At the present time, it is generally believed that these coefficients, too, can be expressed in the form of time-integrals of current correlation functions. The evaluation of a current correlation function formula is by no means straightforward in spite of its compact expression. Various methods of computation including those already mentioned[1, 2, 4–10] have been proposed[14], each being different from others in spirit and degree of sophistication[15].

In this paper it will be demonstrated that in the *bulk limit* where N (particle number) $\to \infty$, Ω (volume) $\to \infty$ while N/Ω remains finite, there exists a rigorous closed equation for the one-body distribution function f. It is done in the following steps. The hierarchy equation of lowest order 2.10 contains an integral of the product of the pair potential v and the two-body distribution function f_2. This integral is analysed in terms of connected diagrams and is shown to be expressible in terms of the one-body distribution function f and initial correlation functions χ. Since the latter, χ, are to be given as an initial condition, the equation obtained is closed with respect to f. It is, however, highly non-linear and non-Markoffian. In obtaining this closed equation, no approximations other than those which can be justified in the bulk limit, are introduced.

From the closed equation one can derive a generalized Boltzmann equation, which is closed in f, which can rigorously describe linear and non-linear transport coefficients, and which no longer depends on the initial condition. This means in particular that Bogoliubov's conjecture of the closure in f is in fact correct although his two explicit assumptions mentioned earlier for achieving this closure are not. The analysis in terms of connected diagrams is also useful in the practical calculation of transport coefficients. In fact it has been shown earlier[16] that it allows one to develop the formal density expansion of a transport coefficient in an unambiguous manner. Unfortunately this density expansion is in general divergent, see below. A serious restrictive feature of the connected diagram analysis is that it applies only to a system obeying classical statistics. Although a gas of monatomic molecules in a certain temperature range should fall in this category, systems of great interest such as an electron gas at high density which obeys Fermi–Dirac statistics, cannot be treated by the present method. This shortcoming can be overcome by working with double-time Green's functions $(g^>, g^<)$ in place of the single-time one-body distribution function f. The existence of a closed set of equations for $(g^>, g^<)$ which rigorously describe transport coefficients can be established[17], but will not be discussed in the present paper. The divergence difficulty of the density expansion of a

transport coefficient mentioned earlier can be also overcome by working with Green's functions, which will be discussed elsewhere.

Obviously, the essential point in the present theory is the demonstration of the existence of a closed evolution equation. Previously this was done for a system of interacting quantum molecules obeying classical statistics[18]. The same technique can be extended to a classical imperfect gas. Since the technique is rather involved and since the classical system can be handled with less conceptual complexities, we shall review the essential steps of the demonstration for a classical gas.

2. B–B–G–K–Y HIERARCHY

Let us consider a system of particles interacting with pair forces, characterized by the Hamiltonian

$$H \equiv \sum_j \frac{1}{2m} p_j^2 + \lambda \sum_{j>k} v(\mathbf{r}_j - \mathbf{r}_k)$$
$$\equiv \sum_j h_0^{(j)} + \lambda \sum_{j>k} v^{(jk)}$$
$$\equiv H_0 + \lambda V \tag{2.1}$$

where h_0 and v are respectively the kinetic energy and pair potential energy. The one-body and two-body distribution functions will be defined by:

$$f(\mathbf{rp}, t) \equiv (N!)^{-1} \iint (\Pi \, d^3 r_j)(\Pi \, d^3 p_j) \hat{f}(\mathbf{rp}) \rho(\mathbf{r}_1 \mathbf{p}_1, \mathbf{r}_2 \mathbf{p}_2, \dots, t)$$
$$\text{Tr}\{\hat{f}(\mathbf{rp}) \rho\}$$
$$f_2(\mathbf{rp}, \mathbf{r'p'}, t) \equiv \text{Tr}\{\hat{f}_2(\mathbf{rp}, \mathbf{r'p'})\} \rho(t) \tag{2.2}$$

$$\hat{f}(\mathbf{rp}) \equiv \frac{1}{\Omega} \sum_j \delta^{(3)}(\mathbf{r} - \mathbf{r}_j) \, \delta^{(3)}(\mathbf{p} - \mathbf{p}_j) \equiv \sum_j \hat{f}^{(j)}(\mathbf{rp})$$

$$\hat{f}_2(\mathbf{rp}, \mathbf{r'p'}) \equiv \frac{1}{\Omega^2} \sum_{j \neq k} \delta^{(3)}(\mathbf{r} - \mathbf{r}_j) \, \delta^{(3)}(\mathbf{p} - \mathbf{p}_j) \, \delta^{(3)}(\mathbf{r'} - \mathbf{r}_k) \, \delta^{(3)}(\mathbf{p'} - \mathbf{p}_k) \tag{2.3}$$

$\rho(t)$ is the A-body distribution function which obeys the Liouville equation

$$i \frac{\partial \rho(t)}{\partial t} = i \sum_k \left\{ \frac{\partial H}{\partial \mathbf{r}_k} \cdot \frac{\partial}{\partial \mathbf{p}_k} - \frac{\partial H}{\partial \mathbf{p}_k} \cdot \frac{\partial}{\partial \mathbf{r}_k} \right\} \rho(t)$$
$$\equiv \mathscr{H} \rho(t) \tag{2.4}$$

Here the operator denoted by $\mathscr{H}$ will be called a *Liouville operator*; it is a differential operator which is generated from a given Hamiltonian H and which is convenient in the description of the time development of the system.

Differentiating $f(\mathbf{rp}, t)$ with respect to t and using (2.4), one obtains

$$\frac{\partial f(\mathbf{rp}, t)}{\partial t} = -i Tr\{\hat{f}(\mathbf{rp}) \mathscr{H} \rho(t)\}$$
$$= -i \sum_j Tr\{\hat{f}^{(j)}(\mathbf{rp}) h_0^{(j)} \rho(t)\} - i\lambda \sum_{j \neq k} Tr\{\hat{f}^{(j)}(\mathbf{rp}) v^{(jk)} \rho(t)\} \tag{2.5}$$

where the script $(\mathscr{h}_0, \mathscr{v})$ are the Liouville operators corresponding to (h_0, v), e.g.

$$h_0^{(j)} \equiv i \sum_k \left\{ \frac{\partial h_0^{(j)}}{\partial \mathbf{r}_k} \cdot \frac{\partial}{\partial \mathbf{p}_k} - \frac{\partial h_0^{(j)}}{\partial \mathbf{p}_k} \cdot \frac{\partial}{\partial \mathbf{r}_k} \right\}$$

$$\equiv -i \frac{\mathbf{p}_j}{M} \cdot \frac{\partial}{\partial \mathbf{r}_j}. \tag{2.6}$$

The first term in the third member of 2.5 can be written as

$$-i \sum_j \mathrm{Tr}\{\hat{\mathrm{f}}^{(j)}(\mathbf{rp})\, h_0^{(j)} \rho(t)\} = -\frac{\mathbf{p}}{M} \cdot \frac{\partial}{\partial \mathbf{r}} \mathrm{f}(\mathbf{rp}, t)$$

$$\equiv -i h_0(\mathbf{rp})\, \mathrm{f}(\mathbf{rp}, t) \tag{2.7}$$

In a similar manner the second term can be written as

$$-i\lambda \sum_{j \neq k} \mathrm{Tr}\{\hat{\mathrm{f}}^{(j)}(\mathbf{rp})\, v^{(jk)}\rho(t)\}$$

$$= -\lambda \iint \mathrm{d}^3 r_2\, \mathrm{d}^3 p_2 \frac{\partial v(\mathbf{r} - \mathbf{r}_2)}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{p}}\, \mathrm{f}_2(\mathbf{rp}, \mathbf{r}_2\mathbf{p}_2, t)$$

$$\equiv -i\lambda \mathrm{tr}^{(2)}\{v^{(12)}\mathrm{f}_2^{(12)}(t)\} \tag{2.8}$$

where the symbol $\mathrm{tr}^{(2)}$ means the integration with respect to the phase-space variables of the second particle. Therefore one can rewrite 2.5 as

$$\left[\frac{\partial}{\partial t} + i h_0(\mathbf{rp})\right] \mathrm{f}(\mathbf{rp}, t) = -i\lambda\, \mathrm{tr}^{(2)}\{v^{(12)}\, \mathrm{f}_2^{(12)}\} \tag{2.9}$$

This equation can be written in a more familiar fashion

$$\left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{M} \cdot \frac{\partial}{\partial \mathbf{r}}\right] \mathrm{f}(\mathbf{rp}, t) = \iint \mathrm{d}^3 r' \mathrm{d}^3 p' \frac{\partial v(\mathbf{r} - \mathbf{r}')}{\partial \mathbf{r}} \cdot \frac{\partial \mathrm{f}_2(\mathbf{rp}, \mathbf{r'p'}, t)}{\partial \mathbf{p}}. \tag{2.10}$$

This is the lowest-order equation of the B–B–G–K–Y hierarchy. The notations, more abstract and more concise than usual, which were used in the derivation of 2.10 will be found convenient in the following development of the theory.

3. CLOSED EVOLUTION EQUATION FOR DISTRIBUTION FUNCTION f

The Hamiltonian H and therefore the corresponding Liouville operator $\mathscr{H}$ are independent of time. The formal solution to (2.4) is

$$\rho(t) = \exp(-it\mathscr{H})\, \rho(0) \equiv \exp(-it\mathscr{H})\, \rho \tag{3.1}$$

In general a function of an operator is defined as a polynomial or a power series, e.g.

$$\exp(-it\mathscr{H}) \equiv 1 - it\mathscr{H} + \tfrac{1}{2}(-it)^2\mathscr{H}^2 - \dots \tag{3.2}$$

This operator $\exp(-it\mathscr{H})$ is a function of the coupling parameter λ, and can be expanded in a power series of λ (perturbation series):

$$\exp\left(-it\mathscr{H}\right) = \exp\left(-it\mathscr{H}_0\right) \{1 + \sum_1^\infty (-i\lambda)^k \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{k-1}} d\tau_k$$

$$\times \; \mathscr{V}(\tau_1)\mathscr{V}(\tau_2)\dots\mathscr{V}(\tau_k)\} \tag{3.3}$$

$$\mathscr{V}(t) \equiv \exp\left(it\mathscr{H}_0\right) \mathscr{V} \exp\left(-it\mathscr{H}_0\right) \tag{3.4}$$

A many-body distribution function ρ to be specified at the initial time $t = 0$ contains all the information about the inhomogeneity and particle correlation of the system at $t = 0$. The latter are, however, more conveniently described by the reduced distribution functions f, $f_2, \dots$, which may be defined through 2.2 with the use of time-independent ρ. The particle correlation can be more appropriately described by the *correlation functions* defined by:

$$f^{(1)} \quad \equiv f(\mathbf{r}_1\mathbf{p}_1) \equiv \chi(\mathbf{r}_1\mathbf{p}_1) \equiv \chi^{(1)}$$

$$f_2^{(1,\,2)} \equiv f_2(\mathbf{r}_1\mathbf{p}_1, \mathbf{r}_2\mathbf{p}_2) \equiv \chi^{(1)}\chi^{(2)} + \chi_2(\mathbf{r}_1\mathbf{p}_1, \mathbf{r}_2\mathbf{p}_2)$$

$$\equiv \chi^{(1)}\chi^{(2)} + \chi_2^{(1\,2)}$$

$$f_3^{(1\,2\,3)} \equiv \chi^{(1)}\chi^{(2)}\chi^{(3)} + \chi^{(1)}\chi_2^{(23)} + \chi^{(2)}\chi_2^{(31)} + \chi^{(3)}\chi_2^{(1\,2)} + \chi_3^{(1\,2\,3)} \tag{3.5}$$

It is clear that one can specify the initial condition by giving $\chi \equiv f, \chi_2, \dots,$ $\dots$ rather than ρ. In fact this specification is obviously more realistic.

Let us now consider

$$- i\lambda \sum_{j \neq k} \text{Tr} \{\hat{f}^{(j)}(\mathbf{rp}) \; v^{(jk)}\rho(t)\}$$

$$= - i\lambda \sum_{j \neq k} \text{Tr} \{\hat{f}^{(j)}v^{(jk)} \exp\left(-it\mathscr{H}\right)\rho\} \tag{3.6}$$

which appeared in 2.5 and which was transformed into an integral involving f_2 in 2.10. We expand $\exp\left(-it\mathscr{H}\right)$ in a perturbation series by means of 3.3, and regard ρ as the N-body reduced distribution function f_N, and expand the latter as

$$f_N^{(1,\,\dots,\,N)} = \prod_1^N \chi^{(j)} + [\chi_2^{(1\,2)} \prod_3^N \chi^{(j)} + \text{(similar terms obtained by permutations)}]$$

$$+ \chi_3^{(1\,2\,3)} \prod_4^N \chi^{(j)} + \dots \tag{3.7}$$

the Nth equation of 3.5. We represent terms in the expansion of 3.6 by diagrams as follows.

We draw N horizontal solid lines for the N particles. The operator $\hat{f}^{(j)}$ is represented by the open circle at the left end of the particle line j, and $v^{(jk)}$ by a vertical dotted line, called a *potential bond*, connecting the pair of particle lines (j, k) at their left ends. Corresponding to

$$v^{(jk)}(\tau) \equiv \exp\left(i\mathscr{H}_0\tau\right) v^{(jk)} \exp\left(-i\mathscr{H}_0\tau\right)$$

$$= \exp\left[i\tau(h_0^{(j)} + h_0^{(k)})\right] v^{(jk)} \exp\left[-i\tau(h_0^{(j)} + h_0^{(k)})\right] \tag{3.8}$$

we draw a potential bond (j, k) at t (time) $= \tau$, where the time is measured from the right to the left. The $\chi_l, l \geqslant 2$, are indicated by broken lines, called *correlation bonds*, connecting the right ends of the particle lines. In this way

we can represent all the expansion terms in one-to-one correspondence. Some typical diagrams are drawn in *Figure 1*.

A diagram is said to be *connected* if the set of particles describing the
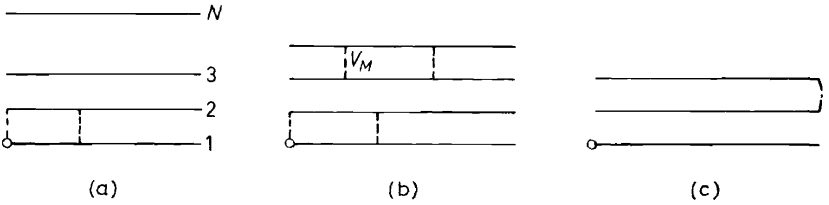
Figure 1. Diagrams representing components of statement 3.6. Diagram a is connected and b and c are disconnected.

potential and/or correlation bonds cannot be separated out into two or more subsets. Otherwise the diagram will be called a *disconnected* one. For example, the diagram b is disconnected.

None of the disconnected diagrams contribute to 3.6. This can be proved with the aid of the following two theorems:

Theorem U Any diagram containing an M-type potential bond contributes nothing.

A *potential bond of M-type* is any bond, like the V_M in diagram b, which sees nothing but the two free particle lines on its left. A particle line segment is said to be *free* if the diagram is broken into two by cutting it. This theorem is proved as follows. Let us suppose that an M-type potential bond connects the pair of lines (j, k) at $t = \tau$. The contribution of the diagram will then contain a factor (see 3.8)

$$\mathrm{tr}^{(j)}\mathrm{tr}^{(k)}\{\exp\left[i\tau h_0^{(j)} + i\tau h_0^{(k)}\right] v^{(jk)}\, g^{(jk\cdots)}\} \tag{3.9}$$

where g is a certain function of the variables corresponding to the particles (j, k) and possibly others. This quantity 3.9 can be decomposed into vanishing integrals of the following three types:

$$\mathrm{tr}^{(j)}\{h_0^{(j)}g_1^{(jk\cdots)}\} \equiv \iint\mathrm{d}^3 r_j\mathrm{d}^3 p_j h_0^{(j)}g_1(jk\ldots) = 0$$
$$\mathrm{tr}^{(k)}\{h_0^{(k)}g_2^{(jk\cdots)}\} = 0$$
$$\mathrm{tr}^{(j)}\mathrm{tr}^{(k)}\{v^{(jk)}g_3^{(jk\cdots)}\} = 0 \tag{3.10}$$

which can be simply shown by integration by parts.

Theorem II Any diagram containing a correlation bond with one or more free lines on its left contributes nothing.

Diagram c contains a correlation bond with a free line and yields a vanishing contribution. This is because such a correlation bond contributes a vanishing factor of the form

$$\mathrm{tr}^{(1)}\{\chi_l^{(1,\cdots l)}\} = 0, \qquad l \geqslant 2 \tag{3.11}$$

which can be in turn proved from the definition 3.5. In fact, from the second equation 3.5, $\mathrm{tr}^{(1)}\{\chi_2^{(12)}\} = \mathrm{tr}^{(1)}\{f_2^{(12)}\} - \mathrm{tr}^{(1)}\{\chi^{(1)}\}\,\chi^{(2)} = n\chi^{(1)} - n\chi^{(1)} = 0$; such proof can be extended to the case of higher $l > 2$. This theorem is valid rigorously in the bulk limit.

A disconnected diagram has, by construction, a potential bond and/or a correlation bond, of the types referred to in Theorems I and II, and yields a vanishing contribution. Disregarding all disconnected diagrams, we have only to deal with connected diagrams containing the open circle at $t = t$. We may simplify the drawings by omitting particle lines without potential bonds. For example, we may represent diagram a in *Figure 1* as one component of diagram a in *Figure 2* where in addition we leave out indices for particle lines. By such an *unindexed diagram* we shall imply a collection of particle-indexed diagrams of the same structure.

The power of these two theorems is not limited to the elimination of the disconnected diagrams. In fact it allows elimination of a large number of

(a)　　　　　　　(b)　　　　　　　(c)
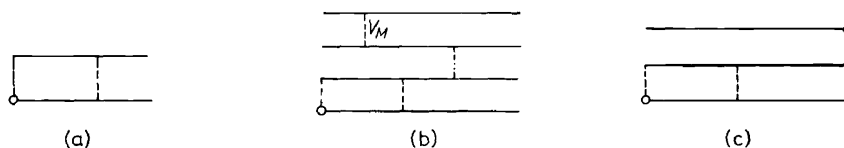
Figure 2. The unindexed diagram a represents the collection of indexed diagrams of the same structure as that of diagram a in *Figure 1*. Connected diagrams b and c do not contribute because they contain an M-type potential and a correlation bond with a free line, respectively.

(a)　　　　　　(b)
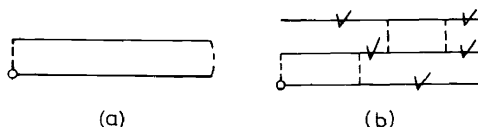
Figure 3. Diagram a contains a d-part and b, two g-parts.

connected diagrams, too. For example, diagrams b and c in *Figure 2* are connected diagrams but they contain respectively an M-type potential bond and a correlation bond with a free line, and thus contribute nothing.

A connected diagram will in general contain several free line segments. Some free segments are indicated by check marks $\sqrt{}$ in *Figure 3*. A diagram will contain a certain number of those parts which consist of non-free line segments, potential bonds and correlation bonds, and which are connected by free segments. Such a part will be called a *d-part* or *g-part* according to whether or not it contains a correlation bond. Diagram a in *Figure 3* has a d-part and diagram b two g-parts.

If a diagram should contain a g-part suspended by two free segments corresponding to the same particle or a d-part standing to the right of a free line segment, it could be *reduced* by suppressing the g- or d- part. Otherwise the diagram is called *irreducible*. In the process of reduction, only the particle line which is marked by the open circle should not be suppressed. With this rule, the reduction becomes unique. Conversely, reducible diagrams can be obtained from an irreducible diagram by *dressing* its free particle lines with g- and/or d- parts.

We have so far considered those diagrams representing 3.6. We may

represent by similar diagrams the expansion of the one-body distribution function

$$f(\mathbf{rp}, t \equiv \sum_j \text{Tr}\{\hat{f}^{(j)} \exp(-it\mathcal{H})\rho\}. \qquad (3.12)$$

The only difference will be that we should omit the potential bond at $t = t$ which appeared in the representation of 3.6. Analysing in a similar manner we can easily see that any diagram giving a non-trivial contribution is a connected one containing a number of g- and/or d- parts, one of which contains the open circle representing the $\hat{f}^{(j)}$. Typical diagrams are drawn in *Figure 4*. It is immediately seen that all such diagrams (except one) are reducible to the unique diagram b in *Figure 4*, which is free from any potential or correlation bond.

(c)   (d)
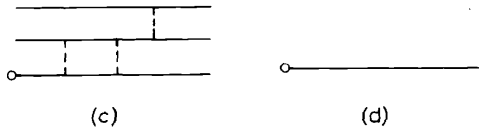
Figure 4, Diagram a, representing a component of $f(\mathbf{rp}, t)$, is uniquely reducible to diagram b. Conversely, diagram a may be obtained from b by dressing the free line with g-parts.

The diagrams drawn here appear to represent the past history of those particles which contribute to the change in $f(t)$ at $t = t$, and the g- and d- parts describe the effects of interaction processes.

Let us consider an irreducible diagram containing a g-part. This contribution can always be expressed in the form of a certain operator g acting on the product of the one-body distribution function corresponding to the system without the interparticle potential ($\lambda = 0$).

$$
\begin{aligned}
f_0(t) &\equiv \text{Tr}\{\hat{f} \exp(-it\mathcal{H}_0)\rho\} \\
&= \sum_j \text{tr}^{(j)}\{\hat{f}^{(j)} \exp(-ith_0^{(j)})\chi^{(j)}\} \qquad (3.13)
\end{aligned}
$$

$$= \sum_\gamma (\overset{m(\gamma)}{\underset{2}{\Pi}} \text{tr}^{(j)}) g(\gamma) \overset{m(\gamma)}{\underset{1}{\sum}} f_0^{(k)} \equiv \Pi g f_0 \qquad (3.14)$$

where $m(\gamma)$ is the number of free lines at the right of a chosen irreducible structure γ.

For example, the contribution of the irreducible diagram a in *Figure 2* can be written down as

$$(-i\lambda)^2 \sum_{jk} \text{tr}^{(k)}\{v^{(jk)} \exp[-it(h_0^{(j)} + h_0^{(k)})] \int_0^t d\tau v^{(jk)}(\tau_1) \exp(i\tau_1 h_0^{(j)})$$
$$\times \exp(i\tau_1 h_0^{(k)}) f_0^{(j)}(\tau_1) f_0^{(k)}(\tau_1)\} \qquad (3.15)$$

Consider now a reducible diagram which upon reduction gives rise to an irreducible diagram. The former can be constructed from the latter by dressing the free lines on the RHS.

By construction the two sets of particles involved in the evolution of any two of the originally free lines are separated from each other. Furthermore,

the structures of all those subdiagrams which upon reduction give rise to a free line can be seen to be identical with the structures of all the connected diagrams for $f(t)$ in the bulk limit. In this identification it is important to notice that the dressing of the free-particle line should be made always to the right, i.e. in the direction of decreasing time since dressing made otherwise would necessarily introduce an M-type potential as seen in the diagram b in *Figure 2*, and therefore would give no contribution. These analyses lead us to write for the contribution of all the irreducible diagrams containing g-parts and the reducible diagrams generated from them

$$\sum_{\gamma} \left(\prod_{2}^{m(\gamma)} \mathrm{tr}^{(j)} \right) g(\gamma) \prod_{1}^{m(\gamma)} f^{(k)} \equiv \sum g \Pi f \tag{3.16}$$

which is obtained by simply replacing every f_0 in 3.14 with f and by summing over all irreducible diagrams.

The irreducible diagrams containing d- parts and the reducible diagrams generated from them can be analysed in a similar manner. Their contribution may be symbolically written as

$$\sum d \, \Pi \, f \, \Pi \, \chi \tag{3.17}$$

which may or may not contain the factors in f but must include one or more of the initial correlation functions χ_l, $l \geqslant 2$.

Thus, we obtain for 3.6

$$- i\lambda \sum_{j \neq k} \mathrm{tr}^{(j)} \mathrm{tr}^{(k)} \{ \hat{f}^{(j)} v^{(jk)} \rho(t) \} = \sum g \Pi f + \sum d \Pi f \Pi \chi \tag{3.18}$$

Using this and (2.7), we can rewrite (2.5) as

$$\left(\frac{\partial}{\partial t} + i h_0 \right) f(t) = \sum g \Pi f + \sum d \Pi f \Pi \chi \tag{3.19}$$

This is an evolution equation which holds rigorously in the bulk limit. Since the correlation functions χ are to be given as the initial condition, this equation is a closed equation with respect to the distribution function f in contrast to the hierarchy equation 2.5 from which it is derived. The equation has, however, infinitely many terms, most conveniently defined in connected diagrams; and it is non-linear and non-Markoffian.

ACKNOWLEDGEMENTS

REFERENCES

[1] N. Bogoliubov, *Problemi Dynamitchesky Theorie Statisticheskey Physike*. OGIS: Moscow (1946); English translation by E. K. Gora, *Studies in Statistical Mechanics*, Vol. I, p. 1. Edited by J. H. de Boer and G. E. Uhlenbeck. North Holland Publ. Co.: Amsterdam (1962).

[2] J. Yvon, *La Théorie Statistique des Fluides et l'Equation d'Etat.* Hermann: Paris (1935).
J. G. Kirkwood, *J. Chem. Phys.* **14**, 180 (1946) and **15**, 72 (1947).
M. Born and H. S. Green, *General Kinetic Theory of Liquids.* Cambridge University Press: London (1949).

[3] D. Bohm and D. Pines, *Phys. Rev.* **92**, 609 (1953).

[4] R. Brout, *Physica*, **22**, 509 (1956).
R. Brout and I. Prigogine, *Physica*, **22**, 621 (1956).

[5] L. Van Hove, *Physica*, **21**, 517 (1955) and **23**, 441 (1957).

[6] W. Kohn and J. M. Luttinger, *Phys. Rev.* **108**, 590 (1957) and **109**, 1892 (1958).

[7] I. Prigogine, *Non-Equilibrium Statistical Mechanics.* Interscience: New York (1962).

[8] R. Balescu, *Statistical Mechanics of Charged Particles.* Interscience: New York (1963).

[9] S. Fujita, *Introduction to Non-Equilibrium Quantum Statistical Mechanics.* W. Saunders: Philadelphia (1966).

[10] A. Lenard, *Ann. Phys. (N.Y.)*, **10**, 390 (1960).

[11] R. Kubo, *J. Phys. Soc. Japan*, **12**, 570 (1957).

[12] See also H. Nakano, *Progr. Theor. Phys.* **15**, 77 (1956).

[13] M. S. Green, *J. Chem. Phys.* **22**, 898 (1954).
H. Mori, *J. Phys. Soc. Japan*, **11**, 1029 (1956).
R. Kubo, M. Yokota and S. Nakajima, *J. Phys. Soc. Japan*, **12**, 1203 (1957).
J. A. McLennan Jr, *Phys. Rev.* **115**, 1405 (1959).
J. A. McLennan, *Physics of Fluids* **3**, 493 (1966).
J. M. Luttinger, *Phys. Rev.* **135**, A1505 (1964).
N. Hashitsume and S. Fujita, *J. Math. Phys.* **5**, 1572 (1964).

[14] D. Montgomery and D. A. Tidman, *Plasma Kinetic Theory.* McGraw-Hill: New York (1964).
L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics.* W. Benjamin: New York (1962).
W. E. Brittin (Editor), *Lectures in Theoretical Physics*, 9C: *Kinetic Theory.* Gordon and Breach: New York (1967).
Yu. L. Klimontovich, *Statistical Theory of Non-Equilibrium Processes in a Plasma.* Massachusetts Institute of Technology: Cambridge, Mass. (1967).

[15] For a critical and comparative review, see T. Y. Wu; *Kinetic Equations of Gases and Plasmas.* Addison-Wesley: Reading, Mass. (1966).

[16] S. Fujita, *Proc. Nat. Acad. Sci. Wash.* **56**, 16 (1966).

[17] S. Fujita, *J. Phys. Soc. Japan*, **27**, 1096 (1969).

[18] S. Fujita, *J. Phys. Soc. Japan*, **26**, 505 (1969).